THE AMERICAN MINERALOGIST, VOL. 50, JANUARY–FEBRUARY, 1965

THE SAMPLING ERROR IN MODAL ANALYSIS

M. B. BAYLY,[1] *Department of the Geophysical Sciences, University of Chicago, Chicago, Illinois.*

ABSTRACT

Apart from errors arising in the analytical process, an unavoidable sampling error arises in modal analysis as soon as an area is selected as representative of the rock. The variance V_s of the distribution of this error was estimated (by replication) in twenty analysis situations. The straight line that fits the twenty estimates with least squared deviations is

$$V_s = 132pi^2/A$$

where

V_s = sampling variance,
p = estimated percentage present of the mineral estimated in each analysis situation,
i = mean intercept cut off by random lines on the mineral estimated,
A = sample area.

The standard deviation of samples approached 2% for quartz even for sample areas as large as 75 cm², for rocks such as the Wausau granite in which i for quartz = 1.92 mm.

OUTLINE AND APPLICATION

Introduction. The purpose of the study is to find with what precision a modal analysis indicates the percentage of a given mineral that is present in a hand-specimen.

An essential step in the current procedure for modal analysis is the preparation from the hand-specimen of a planar sample area, usually by sawing it. This step immediately separates two components in the discrepancy between an analysis and the actual proportions of the hand-specimen: there is an error in taking the sample area as representing the hand-specimen, and there are subsequently all kinds of errors in analyzing the sample area chosen. The first is conveniently called the Sampling Error and all the others collectively called the Counting Error (because point-counting happens to be the analysis technique used throughout). The two errors are to a first approximation independent, having separate distributions with variances here referred to as the Sampling Variance, V_s, and the Counting Variance, V_c (the exact distribution of the Counting Variance is actually conditional upon the particular value of the Sampling Error that is drawn from the distribution of values of that error on cutting a sample surface). This slight interdependence is discussed later. So, we may expect a series of independent analyses of one hand-specimen to be distributed with a variance V_p where

$$V_p = V_c + V_s$$

[1] Present address: Rensselaer Polytechnic Institute, Troy, New York.

The size of V_c has been extensively investigated by Chayes (1956), who has shown that it may be approximated by $p(100-p)/n$ (p = estimated percentage present of mineral of interest, n = number of points counted). The size of V_s is less well known; the experiments to be described are an attempt to improve our knowledge of its size and its dependence on the analysis conditions.

One condition affecting V_s is the area used as the sample; another condition is the coarseness of the rock; a third condition is the uniformity of the rock's grains. On the basis of general sampling experience, we expect that V_s will fall with increase in the sample area but will rise with increase in the rock coarseness and with increase in grain variability.

The purpose of the study can now be stated more precisely: it is to find quantitatively how the sample area and the rock coarseness affect precision; no attempt is made to find how the grain variability affects precision, directly. However, it turns out that grain variability is a factor in the relation between rock coarseness and precision—the last two are inversely proportional in rocks of equal variability but not otherwise—so that grain variability has to be considered even to satisfy the limited aims stated. The consideration is given later.

The conclusion is that a realistic mathematical model, at least of the rocks used in this test, can be constructed using the assumption that all are equally variable. The model then suggests that the relation of precision to sample area and rock coarseness will be

$$V_s = kPi^2/A \quad \text{or} \quad V_s \cong kpi^2/A \tag{1}$$

where

V_s = sampling variance
P = actual percentage present of mineral of interest
p = estimated percentage present of mineral of interest
i = mean intercept cut off by a random line across a grain of the mineral of interest
A = sample area
k = a constant applicable to the rocks used in this test and to all others texturally similar.

If a separate suite were found, again all similar to each other in variability but, as a suite, more variable than the suite here tested, the equation for V_s would have the same form but a different value for k.

In earlier papers (Chayes, 1956; Bayly, 1960 a, b), behavior was discussed in terms of the IC number for the rock, that is, the number of major mineral changes encountered in unit length of a straight traverse. This number rises as the mean intercept falls and accordingly the earlier

hypothesis was

$$V_s = k'P/A \cdot IC^2$$

The reason for abandoning IC in favor of the mean intercept i is that i can be measured for a single mineral; IC as defined (Chayes, 1956) is at best an average for all major minerals in the rock, and is consequently an uncertain index of the grain-size of a single mineral.

The remainder of the text discusses a series of replicate analyses whose results were tested to answers the questions

1. are the results compatible with the hypotheses embodied in Equation 1?
2. are departures from Equation 1 sufficiently systematic to suggest any alternative hypothesis?
3 what is the value of k?

The analysis program is described under "Procedures." The quantities of interest, V_s and $\bar{p}i^2/A$, are shown graphically in Figures 1 and 2. (Hereafter, $\bar{p}$ is used for the best available estimate of P, formed by averaging all analysis results on a given mineral, while p is retained to refer to any single analysis result.)

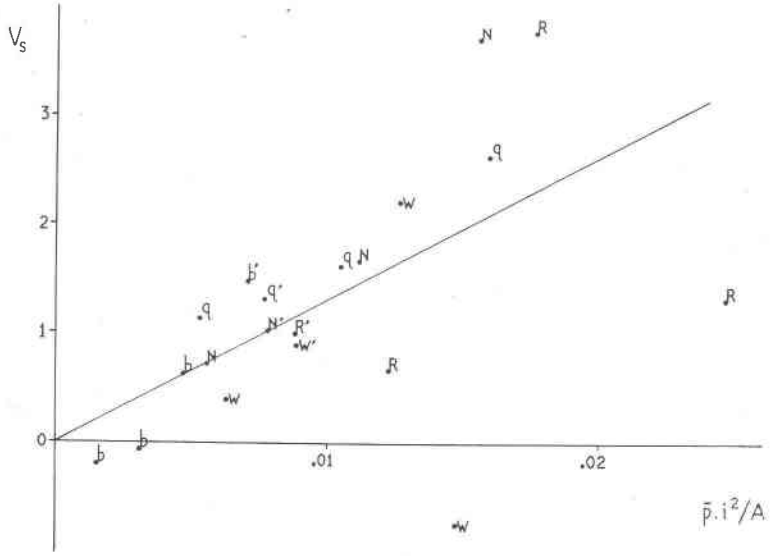

FIG. 1. Plot of V_s against $\bar{p} \cdot i^2/A$. The letter symbols show what mineral was studied to generate each point, thus: b: biotite in "Platinum Grey" granite (commercial name); q: quartz in "Platinum Grey" granite; W: potash feldspar in Westerly granite, R. I.; N: quartz in "North Star" granite (commercial name); R: quartz in Wausau granite, Wisconsin. The line that is eventually accepted as the best fit to the points (as regards minimum squared deviations), with a gradient of 132, is shown.

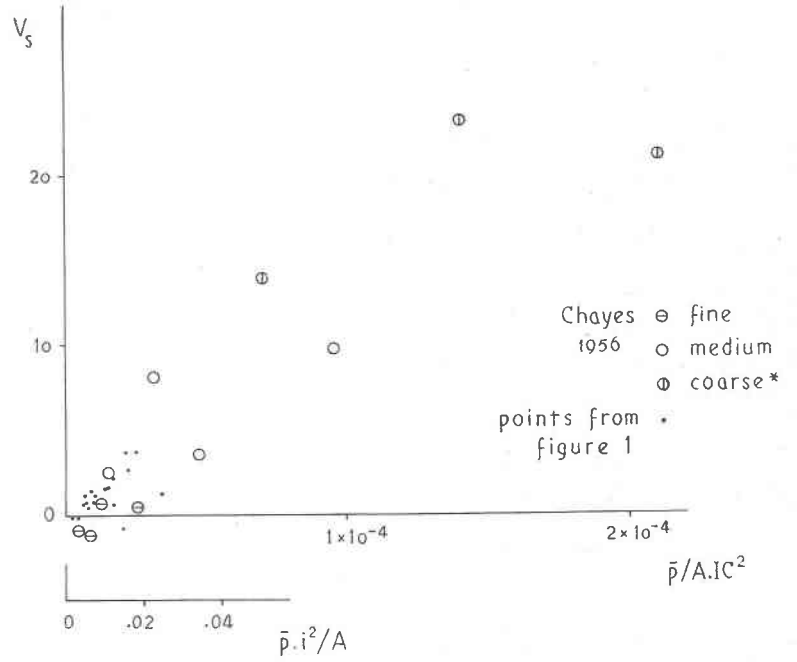

FIG. 2. The data of Fig. 1 are compared with those of Chayes (1956). Because the data involve two different coarseness indicators, i and IC, between which the constant of proportionality is not known, the horizontal scales are not necessarily in correct proportion. Nevertheless, the superposition shows up the difference in range of the two tests.

(Chayes' data were treated as follows. For each block on p. 66, V_c and so V_s were calculated using

$$\bar{p} = \Sigma\bar{x}/3 \qquad \hat{V}_c = \bar{p}(100 - \bar{p})/n \qquad V_s = \overline{V_p} - \hat{V}_c$$

($\bar{x}$, $\overline{V_p}$ and n are given on p. 66)

For abscissae, the products $p/A\cdot IC^2$ were used, since rocks whose textures are ideally similar to one another give Chayes' coarseness index IC inversely proportional to i used here; that is,

$$\bar{p}/A\cdot IC^2 \propto \bar{p}\cdot i^2/A$$

* The coordinates of the fourth point in this set are $(4.22\times10^{-4}, 98.4)$.

Inspection of Fig. 1 suggests, firstly, that the results are too erratic to be sensitive indicators of any departure from the expected form; and secondly, that they are, within their scatter, compatible with that form. A numerical test presented further on confirms their compatibility and shows that, for least sum of squared errors, $k = 132$.

Application. In any actual analysis, n and A are at the operator's discretion. For p, there are two situations. In one, the analysis has been performed and an estimate of its precision is required in retrospect; in this case, a value of p is provided by the analysis result. In the other, it is desired to use the formulae as a guide to suitable values of n and A before the analysis is begun; in this case a rough estimate of p must be made and used.

In either case, a coarseness indicator is required. It cannot be claimed that either i as used here or the IC of previous work is satisfactory. In determining either, the operator is faced with many decisions over specks which may or may not be counted as grains; consequently, the numerical value assigned to the rock is bound to be affected by the magnification used and other operator qualities. Moreover, since the indicator is squared in Equation 1, the effect of the uncertainty is particularly marked. On this account, it was thought that any opportunity to improve the precision of the coarseness measure should be taken in the experimental tests, and accordingly i was measured in preference to IC. (In the present experiments, individual intercepts were measured, because their distribution was of interest as well as their mean value. Normally, i would be estimated as (length of a traverse)$\times$p/N, where N = number of intercepts made during the traverse on the mineral of interest. The first estimate of i, based on a rough estimate of p, would be refined as soon as an accurate measurement of p was made.)

To return to the procedure in a hypothetical case where a mean intercept has been determined: in this case V_s and V_e are both calculable for any values of n and A. If a desired value for the combined variance V_p has been decided in advance, an operator can calculate a variety of pairs of n and A, any pair of which will generate the required V_p. Between such pairs, the choice is made on the relative costs of counting points and preparing sample areas. Numerical examples of such calculations and decisions have been given (Bayly, 1960b) so that here one more example is enough:

if P = 30% and n = 1000, $V_e = 30.70/1000 = 2.1$; also, if i = 1.0 mm and A = 2500 mm², $V_s = 132.30.1/2500 = 1.5$; then $V_p = V_e + V_s = 3.6$ and the estimated standard deviation of a result $= \sqrt{3.6} = 1.9\%$.

To conclude the non-statistical discussion of the results, an observation is made directly from Fig. 1. The five points marked with dashes all represent analysis sets which *in practice* had sampling variances between .75 and 1.5. The areas analysed were as follows:

b	71 mm²
W	152 mm²
g	1189 mm²

N	4231 mm²
R	14970 mm²

The importance of adjusting the area analysed to the coarseness of the rock is clear; but the suggestion that coarse rocks are less satisfactory material for analysis than fine ones receives no support.

Details of Procedures

Experimental arrangements. Four rocks were used, ranging in coarseness from the Westerly granite (the finest) to the Wausau granite, Wisconsin, (the coarsest). From these, five sets of experimental slabs were prepared, one rock furnishing two sets. Each set of slabs provided 96 working areas —24 large, 24 medium, 24 small and 24 very small; thus there were twenty batches of 24. Each area was analysed by point-counter in a rudimentary way, *i.e.* one unmistakable mineral was chosen and the rock was analysed into "chosen mineral" and "the rest"—for example, in the Westerly granite, the potash feldspar alone was estimated. Each batch of 24 similar analyses gave a variance about the mean of the batch, characteristic of the mineral's coarseness, its amount and the area analysed. The arrangements are shown in Table 1 and the counts, areas and results in Table 2.

It was suggested in the Introduction that the experimental result V_p has two components, V_s and V_c, where V_c is approximated by $\bar{p}(100-\bar{p})/n$. Accordingly, the first operation on the results is to estimate V_c for each batch (Table 2, col. v) and so, by subtraction, V_s (Table 2, col. vi). A plot of V_s against the appropriate product $\bar{p}\cdot i^2/A$ (Table 2, col. viii) is given in Fig. 1.

An apparent anomaly exists in Table 2, in that column vi contains three negative values. It would be absurd to suggest that the sampling variance V_s actually has a negative value. However, since the term $\bar{p}(100-\bar{p})/n$ is at best an approximation for the counting variance V_c, it is not absurd to reach negative values for some of the subtractions. The negative values are compatible with what is already recognised, that $\bar{p}(100-\bar{p})/n$ if anything overestimates V_c (Chayes, 1956, p. 91). It must be borne in mind that the experiment sets out basically to discover by how much the total variance V_p exceeds the approximation $\bar{p}(100-\bar{p})/n$; it is only as an aid in constructing a suitable hypothesis that the excess is discussed in relation to V_s. If any part of column vi requires emendation, therefore, it should be the descriptive heading rather than the numerical values.

Theory of factors affecting precision. We begin by considering the sampling of a hypothetical model surface which resembles a cut rock only in certain respects.

TABLE 1. LAYOUT OF ANALYSES

Rock	"Platinum Grey"[1]		Westerly, Rhode I.	"North Star"[1]	Wausau, Wisconsin
Mineral estimated	biotite	quartz	potash feldspar	quartz	quartz
Label assigned to batch of analyses[2]	b	q	W	N	R
Percentage present $\bar{p}$[3]	9.66%	28.0%	36.4%	29.5%	35.8%
Surface	polished slab	etched polished slab	etched stained thin section	etched polished slabs	
Lighting	ordinary light	reflected beam	transmitted light	reflected beam	
Viewer	petrological microscope ×50	binocular microscope ×10	petrological microscope ×50	simple lens ×5	
Point system	mechanical stage	overlaid grid	mechanical stage	overlaid grid	
Mean intercept on mineral, $\bar{i}$[4]	.228 mm	.572 mm	.193 mm	1.06 mm	1.92 mm

[1] Commercial names.

[2] For details of the analyses constituting each batch, see Table 2.

[3] Grand mean of all analyses in batch.

[4] The intercepts were measured in a separate experiment, in which a grid of lines was superimposed on an image of the rock section. The magnification permitted direct measurement of the intercepts with a scale. Each mean is based on 400 measurements.

The spacing was such that many grains were not cut at all and very few grains were large enough to be cut by two successive grid lines. However, there were many grains which, by being re-entrant or by carrying inclusions, gave two or more intercepts on one grid line. Inclusions or re-entrants measuring less than 1/20 of the intercept they interrupted were disregarded.

The model consists of an infinite population of grain-sections of the mineral of interest, whose areas are a where a is a random variable with mean and variance μ_a, σ_a^2. These grain-sections are distributed over an infinite number of rectangles each of area A, in such a manner that, if m is the number of grain-sections in a rectangle, m is a random variable with mean and variance μ_m, σ_m^2. In the model, the grain-sections occupy a sufficiently small proportion of each rectangle for neighboring grains not to interfere *i.e.* for the separate a's within a rectangle to be independent and for m to be independent of the a's. (The remainder of each rectangle is perhaps conveniently imagined as a featureless paste.)

Let a rectangle selected at random contain m grain-sections whose total area is t. That is:

$$t = \sum_{1}^{m} a_j$$

TABLE 2. ANALYSIS RESULTS

(i) Batch Label	(ii) Count Length n	(iii) Sample Area A	(iv) V_p	(v) $\hat{V}_e$	(vi) V_s	(vii) $\sigma(V_s)$	(viii) $\bar{p}\cdot i^2/A$
b	700	71.1 mm²	2.70	1.23	1.47	.763	70.7×10^{-4}
	700	106.7	1.91	1.23	0.68	.540	47.0
	700	160	1.16	1.23	−0.07	.328	31.3
	700	320	1.03	1.23	−0.20	.292	15.7
q	660	575 mm²	5.73	3.05	2.68	1.62	160×10^{-4}
	1000	871	3.62	2.01	1.61	1.02	105
	1350	1189	2.80	1.49	1.31	.791	76.8
	2000	1743	2.14	1.00	1.14	.605	52.5
W	600	91.5 mm²	3.11	3.86	−0.75	.880	148×10^{-4}
	700	106.7	5.53	3.32	2.21	1.56	127
	1000	152	3.20	2.31	0.89	.905	88.8
	1400	213	2.05	1.66	0.39	.580	63.4
N	720	2120 mm²	6.60	2.89	3.71	1.86	156×10^{-4}
	1008	2960	3.72	2.06	1.66	1.05	112
	1440	4230	2.46	1.44	1.02	.698	78.2
	2016	5920	1.75	1.03	0.72	.495	55.9
R	720	5350 mm²	4.49	3.18	1.31	1.27	247×10^{-4}
	1008	7490	6.06	2.28	3.78	1.71	177
	1440	10690	2.25	1.59	0.66	.637	123
	2016	14970	2.14	1.14	1.00	.605	88.2

Batch Label : Each batch contains 96 analyses, 24 for each combination of n and A shown.

V_p : Each value of V_p is calculated from 24 analyses by finding their mean, summing the squared deviations from that mean and dividing by 23.

$\hat{V}_e$: Each value of $\hat{V}_e$ is calculated from $\hat{V}_e=\bar{p}(100-\bar{p})/n$. The ornament is intended as a reminder that V_p and $\hat{V}_e$ are formed in different ways: V_p is directly based on replicate measurements, whereas $\hat{V}_e$ is based on a formula that is established independently of the present data; of the present data, only $\bar{p}$ and n are involved in the estimation of $\hat{V}_e$.

V_s : Each value of V_s is calculated as $V_p-\hat{V}_e$.

$\sigma(V_s)$: This quantity is introduced and used only in "Tests of the Results."

where $j=1, 2, \cdots m$ and a_j are the areas of the m grains in the rectangle.

Then, writing $E(t \mid m)$ for the expectation of t given a particular value of m, we have—

$$E(t \mid m) = m\cdot E(a_j) = m\cdot\mu_a \quad \text{and} \quad V(t \mid m) = m\cdot V(a_j) = m\cdot\sigma_a^2,$$

since $V(t|m)$ is the variance of the sum of m independent random variables all drawn from the same distribution. It is absolutely necessary at this point to assume that the separate a's within a rectangle are independent. It is difficult to believe that this can be true of real grains, but no progress has been made with any alternative assumption. The effect of departure from this condition in real rocks is discussed later.

From these we have, first,—

$$E(t) = E[E(t|m)] = E[m \cdot \mu_a] = \mu_a \cdot \mu_m$$

Also by definition of the variance,—

$$V(t|m) = E(t^2|m) - [E(t|m)]^2$$

whence

$$E(t^2|m) = m \cdot \sigma_a^2 + (m \cdot \mu_a)^2$$

and

$$\begin{aligned} E(t^2), \quad &= E[E(t^2|m)] = E[m \cdot \sigma_a^2 + (m \cdot \mu_a)^2] \\ &= \mu_m \cdot \sigma_a^2 + \mu_a^2 \cdot E[m^2] \\ &= \mu_m \cdot \sigma_a^2 + \mu_a^2 \{V(m) + [E(m)]^2\} \\ &= \mu_m \cdot \sigma_a^2 + \mu_a^2 \{\sigma_m^2 + \mu_m^2\} \end{aligned}$$

Squaring $E(t)$ and subtracting, we have

$$E(t^2) - [E(t)]^2, \quad = V(t), \quad = \mu_m \cdot \sigma_a^2 + \mu_a^2 \cdot \sigma_m^2$$

This accords with the informal conception of part of the variance arising from the differing grain areas—the first term—and part arising from the differing number of grain-sections per rectangle—the second term.

We now substitute for μ_a, σ_a and σ_m. Many natural distributions of an infinity of items into an infinity of cells can be shown to approximate the Poisson distribution. This distribution is particularly to be expected where the presence of one item in a cell does not affect the probability of another item occurring in the same cell or its location within that cell. Now these two conditions were implied when the specification was made that the a's within a rectangle are to be independent and that m is to be independent of the a's. It is no extension, therefore, to assume a Poisson distribution for m and to put $\sigma_m = \sqrt{\mu_m}$.

It is next proposed that, if i = mean intercept cut off on a random line by grains of the mineral of interest, then in the five rocks studied, $\mu_a \propto i^2$ and $\sigma_a \propto i^2$. The argument in favour of these proportionalities is as follows: consider a rock section and a photographic enlargement of it by, say, Z diameters. Between these two, there is absolute correspondence in all but size. If the distribution of grains' maximum diameters in the section is $p(d) = f(d)$, then in the photograph $p(Zd) = f(d)$; and if the distribution of grain areas in the section is $p(a) = h(a)$, then in the photograph $p(Za)$

$=h(a)$. Further, the standard deviations of the distributions of random intercepts may be written σ_i and $Z\sigma_i$, and their means i and Zi, while the standard deviations and means of the grain areas in section and photograph may be written σ_a and $Z^2\sigma_a$, and μ_a and $Z^2\mu_a$. The coefficient of variation $\sigma/_{mean}$ is in either case unchanged in passing from rock to photograph.

Of course, in the situation described there is also one-one correspondence in every individual speck and crack between a section and its photograph; but it is possible for the above statistical correspondences to hold between two sections of rock without the one-one correspondence of

TABLE 3. GRAIN INTERCEPT MEASUREMENTS[1]

Mineral Measured	Mean Intercept i	Standard Deviation σ_i	$\sigma_{i/i}$
potash feldspar in Westerly gt.	0.193 mm	0.189 mm	0.98
biotite in "Platinum Grey"	0.228	0.204	0.89
quartz in "Platinum Grey"	0.572	0.542	0.95
quartz in "North Star"	1.06	1.14	1.08
quartz in Wausau gt.	1.92	1.57	0.82

[1] The method of measuring the intercepts is described in Table 1, note [4].

individual grains. Two such sections might fairly be called statistical scale models; and it is suggested here that the five rocks of the experiment approximate statistical scale models of each other. The only factual evidence bearing on this suggestion is given in Table 3, where it is seen that for intercepts, the coefficient of variation is rather constant. If on these slender grounds, the possibility is entertained that the five rocks are, roughly, statistical scale models of each other, then it is natural to substitute $\mu_a = k_1 i^2$, $\sigma_a = k_2 i^2$ where k_1 and k_2 are wholly unknown constants.

The results, introducing these substitutions into the expressions for V(t) and E(t), are

$$V(t) = \mu_m(k_2 i^2)^2 + (k_1 i^2)^2 \mu_m = (k_1^2 + k_2^2) i^4 \mu_m$$

and

$$E(t) = k_1 i^2 \mu_m$$

Combining,

$$V(t) = \frac{k_1^2 + k_2^2}{k_1} \cdot i^2 E$$

Finally, the symbol u is introduced for the percentage of a rectangle which is occupied by the mineral of interest, $= 100t/A$. Then

$$V(u) = \frac{10^4}{A^2} \cdot V(t) \quad \text{and} \quad E(t) = \frac{A}{10^2} \cdot E(u),$$

giving

$$V(u) = 100 \frac{k_1^2 + k_2^2}{k_1} \cdot \frac{i^2 E(u)}{A} \tag{2}$$

This result is now applied to the modal analysis of real rock surfaces. V(u), the variance of the percentage of a sample occupied by the mineral, corresponds to $V_s \cdot E(u)$ corresponds to the true percentage P of the mineral in the hand-specimen. The theoretical result (2) therefore suggests the hypothesis in real terms—

$$V_s = kPi^2/A$$

which has already been given as Equation 1.

Tests of the results. The data of Table 2 and Figure 1 can be tested in three ways. Firstly, the five groups of four points can be tested separately for linearity; secondly, the whole group of twenty points can be tested for linearity; thirdly, the relative merits of one overall line or five separate lines can be examined.

Firstly, a variance is assigned to each value of V_s. V_s was derived from the experimental result V_p by subtraction of the quantity $\hat{V}_c, = \bar{p}(100 - \bar{p})/n$, so that

$$\mathrm{Var}(V_s) = \mathrm{Var}(V_p) + \mathrm{Var}(\hat{V}_c) - 2\,\mathrm{Cov}(V_p, \hat{V}_c)$$

Now the variance of an experimental estimate of a variance whose true value is known, = say V, is $2V^2/f$, where f = degrees of freedom in the estimate, for a variable of Normal distribution (*e.g.* Brownlee, 1960). In these calculations the true variance is not known but V_p has been estimated by the experiment. If an unbiased estimate is substituted for V, a biased estimate of V^2 is likely. A little calculation indicates that such bias is removed by replacing f by f+2. Now V_p is estimated from 24 results from which a mean has been extracted, so that $f = 23$, $f+2 = 25$ and

$$\mathrm{Var}(V_p) = \frac{2}{25} V_p^2.$$

For $\mathrm{Var}(\hat{V}_c)$, consider batch b as an example. In the region around $\bar{p} = 10\%$, $\hat{V}_c, = \bar{p}(100 - \bar{p})/n$, is approximated by

$$(80\bar{p} + 100)/700 = \frac{8\bar{p}}{70} + \frac{1}{7},$$

and $\mathrm{Var}(\hat{V}_c) = 64V_{\bar{p}}/4900$; but $V_{\bar{p}} \cong V_p/96$ so that $\mathrm{Var}(\hat{V}_c)$ is negligible in

comparison with $\mathrm{Var}(V_p)$. The disparity is even more extreme for other batches, so that $\mathrm{Var}(\hat{V}_c)$ and $\mathrm{Cov}(V_p, \hat{V}_c)$ are neglected and we write

$$\mathrm{Var}(V_s) = \mathrm{Var}(V_p) = \frac{2}{25} V_p^2.$$

The values of this expression are shown in Table 2, col. vii, actually in the form of $\sigma(V_s) = \sqrt{\mathrm{Var}(V_s)}$.

Next a regression of V_s onto $\bar{p} \cdot i^2/A$ is calculated for each group of four points. Since for infinite sample area A, the sampling error is zero, a line through the origin is appropriate. Its gradient is calculated using weights (Brownlee, 1960), each value of V_s being weighted by $1/\mathrm{Var}(V_s)$. The five resulting gradients are

(b) 81.6 (q) 175 (w) 24.2 (N) 152 (R) 81.5

The first test consists of finding the deviation of each point from the estimated line for its group. The deviations are squared, weighted and summed and compared with the χ^2 table. Five gradients have been extracted from the twenty results so the remaining degrees of freedom are 15. The resultant sum, 11.3, does not depart from the expected value of $\chi^2(15)$ by a significant amount.

(If the deviations are summed in groups of four and compared individually with $\chi^2(3)$, the sum for the q-group is significantly small, $=0.175$, for which $p<0.025$. However, the possibility of a nonrandom control affecting the q-results alone is rejected; the twenty results are taken as a homogeneous block and so the examination of their separate sums is not appropriate.)

For the second test, a regression of V_s onto $\bar{p} \cdot i^2/A$ is calculated for all twenty points together. The same weights are used, and the resulting gradient is 93.7. The test consists of finding the deviation of each point from this overall line. The deviations are squared, weighted and summed and compared, this time, with $\chi^2(19)$, since only one gradient has been extracted from the twenty results. The sum, 18.0, agrees with the expectation. The hypothesis that the points are randomly distributed about the one overall line is therefore also acceptable.

We now compare a one-line hypothesis with a five-line hypothesis; but first we remark that although the weights used in the foregoing tests are the only ones available for the purpose, the gradients calculated using them are necessarily underestimates. The reason is that, since the experimental values' reciprocals determine the weights assigned, an experimental value below the calculated line receives more weight than a value equally far above the line, on the same abscissa. Once a linear hypothesis is accepted, therefore, it is desirable to reweight the points and recalculate the gradient, using weights determined not by the experimental re-

sults but by the line calculated at first. The resulting gradients are:

overall line	132								
individual lines	(b)	141	(q)	177	(w)	71.2	(N)	163	(R) 109

(For the reason mentioned before, that $Var(V_s)$ is large compared with V_s, the difference from the first estimates is quite large.)

The third test compares the merits of the two hypotheses using these values. The sum of weighted squares about the overall line is 12.5, the weights being calculated using the gradient 132. The sum of weighted squares about the individual lines is 13.5, the weights here being calculated using the five separate gradients quoted. The difference in the weights appropriate under the two hypotheses accounts for the unusual circumstance of the second sum being greater than the first. There are clearly no grounds for preferring the five-line hypothesis.

Discussion. The foregoing sections may be summarized thus:

measurements of intercepts suggest that some simple geometrical assumptions about the rocks studied are acceptable; a theory resting on these assumptions suggests that the results in Table 2 will be distributed about a straight line; tests of the results in Table 2 show that such a straight-line hypothesis is acceptable.

The results do little to show that other hypotheses are less satisfactory, but in this connection, Chayes' results (Fig. 2) are more informative. These results, taken in conjunction with the theoretical argument for a linear relation, make testing against other types of curve unnecessary at the present time. There remain for discussion three statistical questions and the more general question—is the result applicable to other rocks? Taking the last question first, we recall that previously, two constants k_1 and k_2 were introduced to relate μ_n and σ_n to i. Indirect evidence was given that k_1 and k_2 may be near constant over the five rocks tested, but no data about their values in other rocks have come to the author's notice. Only a comparative assertion can therefore be made: errors on rocks whose textures are close to the granites tested will correspond more closely to the hypothesis $k=132$ than rocks whose textures differ markedly from those tested. For example, poikilitic rocks, whose importance was kindly brought to the author's attention by Dr. L. I. Briggs, University of Michigan, require a totally separate study. The three statistical questions are:

1. are m and a independent, as is assumed previously?
2. does $p(100-p)/n$ overestimate V_c?
3. are the Counting Error and Sampling Error uncorrelated as is implied in the equation

$$V_p = V_c + V_s?$$

These are all aspects of the question—does the model discussed resemble

actual rocks? Of course, the purpose of the model is merely to suggest a hypothesis, and once a hypothesis has been found satisfactory in comparison with a highly sensitive block of data, concern over the model which prompted it is unnecessary. In the present work, however, the data are not sensitive and the choice of hypothesis preferred consequently rests heavily on the model generating it.

Regarding the first of these questions, the supposition is made that the rocks involved crystallized from moderately homogeneous fluids. This suggests that in a region where few quartz nuclei existed, for example, growth on each would be heavier than in a region where many nuclei existed. In the terminology adopted, the values of a over a sample rectangle are likely to be large if m for the rectangle is small; that is, m and a are not independent variables. Although this effect is prima facie open to examination in real rocks, the author is not aware of any data on the subject.

The second question is considered by Chayes (1956), who records some data suggesting that V_c is overestimated by the formula used. If V_c is thus overestimated, the points in Figures 1 and 2 should be compared with lines cutting the vertical axis not at the origin but below it; inspection shows that some improvement of fit might result. However, the experimental data are not sufficiently precise to test the question effectively, and it is raised more for the sake of its theoretical relation to question 3.

The binomial model from which $V_c = p(100-p)/n$ is predicted involves n *independent* points *i.e.* n points, each of which has the same probability of falling on, say, quartz. In practice, however, values of n and A which give acceptable precision lead to point spacings such that several points often fall on one grain; that is, runs of quartz-quartz-quartz etc. are more frequent than they would be if every point were truly independent. The effect of this deviation from the model can be seen by imagining the coarseness exaggerated to an extreme degree where there are only, say, two equal grains in the sample area: in this condition, upon the application of a uniform grid of n points, each grain is bound to receive nearly n/2 points and only a few points have a genuine chance of falling on one grain or the other. Thus the variance of the distribution of possible results is much smaller than it would be if all points had a chance to fall on one grain or the other. In short, $V_c < p(100-p)/n$, the inequality being most marked when the ratio of grain size to point spacing is great.

Putting the answers to questions 1 and 2 together leads to the conclusion, with respect to question 3, that V_c and V_s are not absolutely independent. If a sample with abnormally large grains has abnormally few grains, we shall expect

(i) that the Counting Error will be abnormally small, in accordance with the discussion in the preceding paragraph;

(ii) that the Sampling Error will be abnormally large, in accordance with the general finding that a sample with few items is a more erratic indicator of the parent population than a sample with many items.

Conversely, a sample of the same hand specimen that has abnormally many grains is expected to give a slightly larger Counting Error (closer to the full binomial forecast) but a smaller Sampling Error. In other words, a slight negative correlation is to be expected between the magnitudes of the two errors. On the other hand, the sign of one error is wholly independent of the sign of the other. The result is that the two errors, while not independent, are uncorrelated, and no approximation is involved in writing $V_p = V_c + V_s$.

It would not be appropriate to close this discussion without reference to the work of Hasofer (1963) and Solomon (1963). Hasofer's result, quoted and tested experimentally by Solomon, is based on a model involving N spheres of fixed radius R, whose centers are randomly distributed throughout a rectangular prism in such a way that none are cut by the sides. A point-count estimate of p, the percentage of the prism occupied by spheres, is called p_3 and it is shown that

$$\sigma^2(p_3) < 44\frac{pa^3}{RA}\left[1 + 5.8\left(\frac{R}{a}\right)^3\right]$$

where A is as used in this text and a = distance between points in the counting grid.

Since A/a^2 = count length = n of this text, the expression can be rearranged to the form

$$44\frac{p}{n}\frac{(a)}{(R)} + 255p\frac{R^2}{A}.$$

In the derivation of this expression, the counting and sampling variances are not estimated separately, but an independent derivation of the counting variance alone gives a result that corresponds to the first term in all but the constant. Comparing the expression with the formulae discussed above, *viz.*

$$V_p = V_c + V_s = p\frac{(100 - p)}{n} + 132p\frac{}{A}$$

we note very close correspondence in form between the second terms. The difference in form between the two first terms is attributable to Hasofer's assumption of uniform spherical grains versus Chayes' assumption of total independence of all points in the point count. It has been remarked above that where a is small compared with i, all points are not independent; but the dependence in a section through irregular grains is not as great as in a section through uniform spheres. In short, Hasofer's ap-

proach seems likely to over-correct for the dependence neglected by Chayes.

Acknowledgments

I am glad to acknowledge the help of Dr. F. Chayes, who laid the foundations for this study, suggested its pursuance, and kindly furnished the Westerly samples. I also owe a great debt to Dr. W. Kruskal at the University of Chicago: where the treatment is rigorous it is due to his suggestions; where the treatment is not rigorous, it fails in spite of ever-willing attention, criticism and advice. I am also indebted to Dr. B. Trumbo who joined with Dr. Kruskal in discussions of the problem. The work was carried out during tenure of a Rollin D. Salisbury Fellowship at the University of Chicago.

References

Bayly, M. B. (1960a) Errors in point-counter analysis. *Am. Mineral.* **45,** 447–449.

——— (1960b) Modal analysis by point-counter—the choice of sample area. *Jour. Geol. Soc. Australia,* **6,** 119–130.

Brownlee, K. A. (1960) *Statistical Theory and Methodology in Science and Engineering.* John Wiley & Sons, Inc., New York.

Chayes, F. (1956) *Petrographic Modal Analysis.* John Wiley & Sons, Inc., New York.

Hasofer, A. M. (1963) On the reliability of the point-counter method in petrography. *Aust. Jour. Appl. Sci.* **14,** 168–179.

Solomon, M. (1963) Counting and sampling errors in modal analysis by point-counter. *Jour. Petrology,* **4,** 367–382.

Manuscript received, October 7, 1964; accepted for publication, November 6, 1964.