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THE GNOMONIC PROJECTION

CHARLES PALACHE

Harvard University

The crystal is conceived as having a sphere described about its center (sphere of projection), radii of which may be drawn normal to the crystal faces. The points at which the face-normals intersect the sphere are called the poles of the faces and are defined if we know for each two angles: φ the azimuth angle (longitude) referred to a first or zero meridian; and ρ the polar distance (co-latitude).

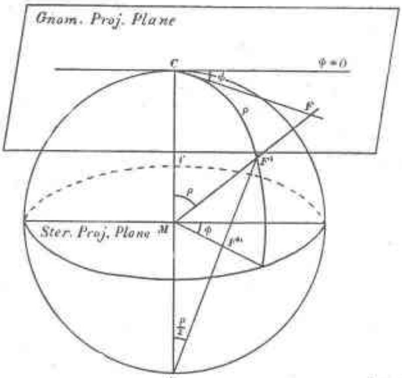
In figure 5 a face-normal is represented by the pole F′, defined by the angles φ and ρ,—angles which can be measured on the two-circle goniometer.

The gnomonic projection is obtained if the face-normals are extended until they intersect a plane tangent to the sphere of projection. The crystal is generally placed with its vertical zone-axis (prism zone) parallel to the vertical diameter of the sphere; the plane of projection is made normal to this and is therefore tangent to the sphere at the north pole and parallel to the plane of the equator. The plane of projection is therefore parallel to the plane of the vertical circle, V, of the two-circle goniometer (see page 25); and to that of the horizontal circle, H, of the contact goniometer (page 46).

PROPERTIES OF THE GNOMONIC PROJECTION[1]

Figure 5 shows in perspective the relations of the gnomonic and spherical projections. It is clear that a face of the crystal to which the radius MF is normal is represented by the points F′ and F on the sphere and plane respectively. Let figure 6 be a

[1] The majority of the figures have been taken, with slight modification, from Boeke's book (see bibliography, 3).



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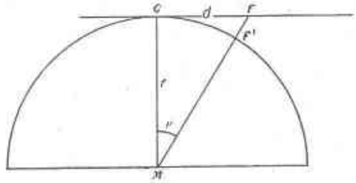

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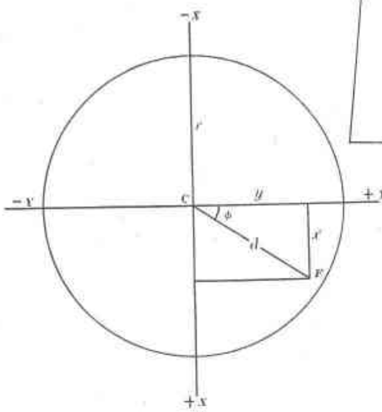

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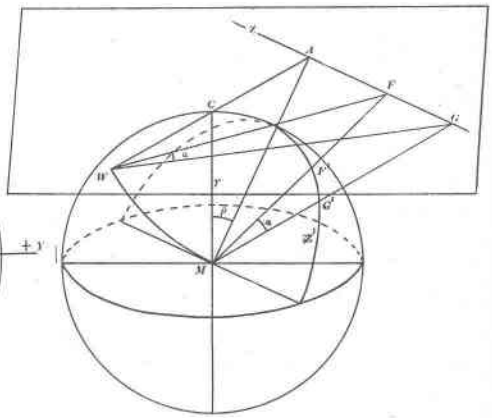

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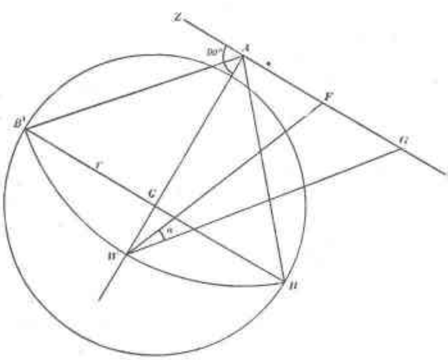

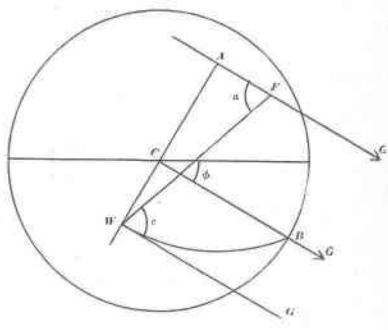

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central section of the upper half of the sphere containing the face normal and the radius MC.

$$\frac{CF}{r} = \tan \rho; \ CF = r \tan \rho; \text{ if } r = 1, \ CF = \tan \rho = d.$$

In general usage r is taken as 5 cm.; therefore the central distance of a face in gnomonic projection, $d = 5 \tan \rho$ cm.

Figure 7 shows in plan the gnomonic projection of the face F of figures 5 and 6. C is the projection of the polar axis and the center of the unit circle with the same radius as the sphere. Let $r = 1$. X and Y are rectangular coördinates thru C. F is the pole of the crystal face whose angles are φ and ρ.

The position of F is determined by its polar coördinates φ and $d = \tan \rho$. It may also be determined by rectangular coördinates:

$$\sin \varphi = \frac{x}{d} = \frac{x}{\tan \rho}, \quad x = \sin \varphi \tan \rho,$$

$$\cos \varphi = \frac{y}{d} = \frac{y}{\tan \rho}, \quad y = \cos \varphi \tan \rho.$$

Considering figure 5 and remembering that projection is by means of radii of the sphere, it will be evident that zones of the crystal which are projected as great circles of the sphere will appear in the gnomonic projection as straight lines. Conversely, the straight line connecting any two gnomonic face-poles represents a zone of the crystal. A crystal face belonging to two zones of the crystal lies at the intersection of two zone lines of the projection. Conversely, the intersection of any two zone lines of the projection is the pole of a possible crystal face.

Poles of faces of the lower half of the crystal are represented by the poles of parallel upper faces.

Faces near the equator have their poles projected to great distances from C; faces on the equator (prism faces) are projected by lines parallel to the plane of projection or to infinity and are defined therefore only by direction lines. This circumstance is the greatest drawback of the gnomonic projection. Working with the usual 5 cm. radius faces with ρ greater than 75° do not appear in a projection of workable size.

The poles of all the faces of a crystal, located by plotting for each φ and $\tan \rho$, or x and y from measurements, constitute the gnomonic projection of the crystal.

Construction of a Gnomonic Projection

The plotting of coördinate angles is done most readily on a sheet provided with a divided circle. Penfield's sheets for stereographic projection serve excellently, a 5 cm. circle being drawn within the divided circle for gnomonic constructions.

Several cases may arise which will be successively described.

1. Given the observed angles of a crystal, v and h; symmetry and the position of the zero meridian of the crystal unknown.

The pole distance, ρ, is found for each face by subtracting its h reading from h_0 (see p. 27.)

The vertical circle of the goniometer corresponds, and is parallel, to the plane of projection. The graduation of this circle may therefore be conceived as superposed upon the divided circle of the drawing sheet. Calling any point on the circumference of the latter zero, say the right hand horizontal diameter, consider the divisions as increasing clockwise to 360°. Selecting now the measurements of any face of the crystal, note its v angle, find the corresponding division of the divided circle, and lay a centimeter scale or a gnomonic protractor with its zero at the center and its edge thru the selected point. It is well to mark this point with a needle until the scale is in position. Now find from the tangent table or from the edge of the protractor the distance corresponding to the angle ρ and prick the point with the needle, circle it lightly, and write beside it the number of the face from the notes. This is done for each face, prism faces being indicated by direction lines with an arrow and the face number. Thus a projection develops on the paper, consisting of the face-poles and prism direction lines. The eye quickly catches in such a projection the alignment of the poles in major zones, which may be drawn with straight-edge and triangle. Symmetry, or the lack of it, is easily established, especially if there are many faces. This method of plotting is rapid and useful for preliminary study of the measurements. It is particularly instructive to beginning students, for it gives an exact picture of the measurements without averaging or adjustments and is almost free from any assumption as to the grouping of the faces. Its main disadvantage is the skew position in which the projection generally lies with reference to the outlines of the drawing sheet, which may be inconvenient when other drawings are to be made on the same sheet.

2. Given φ and ρ. Symmetry known.

With the symmetry known the zero meridian can be established and the value v_0 in terms of the readings on the vertical circle may be determined. One solution for v_0 is graphic by the procedure just described.

The form (010), whose direction line will have $\varphi = 0$ and $v = v_0$, may be present, or it may be defined only by other faces. An example of the calculation of v_0 for a triclinic crystal will be given in a later paper in this series. It can usually be found by averaging pairs of readings of v for prisms or pyramids disposed symmetrically to (010). v_0 being known, it is to be subtracted from the v angles in such a way that the resulting φ angles shall not exceed 180°, plus or minus. This is done by making use of one or other of the relations[1]:

$$+\varphi = v - v_0 \text{ or } (360° + v) - v_0. \quad -\varphi = v_0 - v \text{ or } (v_0 + 360°) - v.$$

By writing v_0 on a slip of paper which can be held just above or beneath the successive angles in the v column a glance will show which of the equations to use and the difference, easily calculated mentally after a little practice, is entered in the φ column to the right. The ρ angles are obtained by subtraction of h from h_0 in the same manner.

It is customary to make the right-hand end of the horizontal line on the paper the zero point for φ. φ is laid off from this point on the divided circle, plus or minus as the case may be and the ρ plotted as before. It is best to draw no zone lines until all poles are plotted. The zones and unit coördinates soon become apparent as plotting proceeds.

3. Plotting without a divided circle.

Draw the unit circle and horizontal and vertical diameters, dividing the circle into quadrants. The φ angle is plotted from a table of chords. Since the table reads only to 45°, when φ is over 45°, its complement, supplement, or $\varphi - 90°$ is taken and plotted from the appropriate quadrantal point. The chord length is taken with the dividers from the scale and transferred to the circle. ρ is then plotted as before.

This method is slow and has no special merit.

4. Plotting from Goldschmidt's *Winkeltabellen*. Polar elements and symbols of the forms given.

This may be done by plotting φ and ρ, as in 2; or by taking $d = \tan \rho$ from the tables; or from the rectangular coördinates,

[1] A rough sketch showing the unit circle with the value of v_0 written at the zero point will be found useful in applying these relations.

x and y, also given in the tables. It is accomplished more rapidly by plotting the three pinacoids and the unit pyramid from their φ and ρ; and drawing parallel coördinates thru this last pole. p_0 and q_0 are thus given graphically. The indices of the forms are transformed so that the third term of each is unity: $\left(\frac{h}{l}\ \frac{k}{l}\ 1\right)$ or (pq1) where p and q are either whole numbers or fractions. (The first two numbers constitute Goldschmidt's symbol.) Each pole is then located by measuring pp_0 and qq_0 with the dividers along the coördinates and drawing parallels to intersection.[1]

5. Plotting from linear axes and interfacial angles.

In the three rectangular systems it is only necessary to lay off the two angles (001) to (011) on the zero meridian; and (001) to (101) on the 90° meridian. These points give the values of p_0 and q_0 and the plotting proceeds from indices as in 4. In the hexagonal system, plotting the angle of a pyramid or rhombohedron face to the basal pinacoid gives the p_0 value. In the monoclinic and triclinic systems the center of coördinates is *not* the center of the circle. While it is easily possible to locate the pinacoid (001) by methods to be described in what follows, it is simpler to transform the linear axes, a and c, to their equivalent polar axes according to the relations to be given later. The plotting is then done as in 4.

Fundamental Constructions of the Gnomonic Projection

1. The angle-point. Measurement of the angle between two faces.

In the perspective figure 8 let F′ and G′ be two face-poles, the angle between which, F′MG′ = α, we wish to measure. The great circle containing F′ and G′ is projected into the straight line Z containing F and G. Imagine the triangle FMG rotated about Z as axis until it comes into the plane of the projection. M, the center of the sphere, will wander to a point W which is on a line CA, perpendicular to Z thru C. AW = AM. The angle FWG = α is the desired angle and can be measured with a protractor or by means of the chord table.

To find W, figure 9. Draw a circle about C with radius r (5 cm), equal to CM of figure 8. Draw CA perpendicular, and

[1]Figure 17 illustrates a general case.

CBB′ parallel, to Z, the zone line containing F and G, two gnomonic face-poles. On the line AC take AW equal AB equal AB′. The construction requires no proof if we note that CB equals CM and that therefore AB = AM = AW.

Definitions.—The circle with center C is called the unit circle (Grundkreis). The perpendicular to the zone line thru C (CA of figures 8 and 9) is called the *central* (Zentrale) of the zone; the point A the zone center.

The point W is called the angle-point (Winkelpunkt) of the zone and is of much importance in many operations in the gnomonic projection. It must be evident from the construction that the angle-point W of any zone will always fall within the unit circle between C and the circumference; and it will lie upon the *central* beyond the center C from the zone line. If Z passes thru C (vertical zone) W will lie on the circumference. If Z is at infinity (prism zone) W will coincide with C.

Special Cases.—1. To measure the angle α between a face-pole, F, and a prism-pole G whose angle φ is given. Figure 10.

G lies at infinity in the direction of CG. Draw the zone line FG parallel to CG and find its angle-point, W. Draw WG′ parallel to FG. The angle FWG′ = WFA is the desired angle α.

2. To measure the angle between two prism poles. The angle is measured at the center between the two direction lines to the prism-poles and is equal to the difference between their φ angles.

3. To measure the angle between two faces having the same φ value (vertical zone). Figure 11.

W lies on the circumference of the unit circle on a radius normal to the zone. Otherwise the construction is the same as in the general case. If one of the poles G′ lies at infinity (prism-pole) its angle to F is given by the angle WFC.

Angular Relation of W, the Angle-Point, to C and A

The relations of W to C and any zone center A may be further explained by consideration of figure 12, a vertical section of the sphere of projection thru a *central* and therefore normal to the zone (compare figures 8 and 9).

Let A be the zone center, W the angle point and ρ the angle AMC which is given in the projection. The triangle AMW is isosceles since AM = AW. The angle θ equals $\frac{1}{2}(90° - \rho)$[1]

[1] The sum of the angles at W and M equals 180° less the angle at A. $180 - (90 - \rho) = 2\rho + 2\theta$. $90 + \rho - 2\rho = 2\theta$. $\theta = \frac{1}{2}(90 - \rho)$.

Now AM = r.sec ρ. Also CA = r.tan ρ; CW = r.tan θ = r.tan $\frac{1}{2}(90° - \rho)$. Therefore AW = r.tan ρ + r.tan $\frac{1}{2}(90° - \rho)$.

Ingenious use of this relation is made by Hutchinson (8) in the use of a gnomonic protractor for rapid determination of the angle-point.

To Find the Gnomonic Pole of a Zone (Edge-pole)

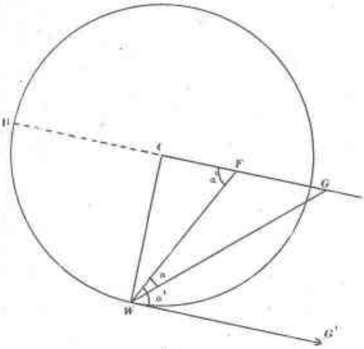
The edge-pole of a zone is the point on the sphere 90° from all points on the zone circle. In the perspective figure 13 P′ is the pole of the zone circle Z′. P the desired gnomonic pole must lie on the *central*, AC, at an angular distance of 90° from A. If the triangle AMP, right-angled at M, be rotated about AP into the plane of the projection, we have the relations of figure 14 and the construction is clear. Draw the *central* CA and CBB′ parallel to Z. Connect A and B and make ABP a right angle. The point P is the desired edge-pole of Z. The construction is more accurate if repeated at B′ as in the figure.

P may be regarded as the point of emergence in the gnomonic projection of the axis of the zone Z; that is of the line normal to it passing thru the center of the sphere. This line is parallel to the edges of intersection of the faces constituting the zone, whence the name, edge-pole, for P, following Hilton. It plays an important part in several operations in the gnomonic projection. P lies between the center of the projection (pole of the prism zone) and infinity (pole of a vertical zone).

Some Relations between the Gnomonic and Stereographic Projections.

In figure 5 the relation between the gnomonic and stereographic projections of the pole F′ is shown in perspective. The two projections are on parallel planes separated by a distance equal to the radius of the sphere, r. In the first the point on the sphere (face-pole) is represented as tho seen from the center, M, of the sphere of projection; in the second the eye-point is at S, the south pole. F″, the stereographic pole of F′, is the point in which the line F′S intersects the equatorial plane.

It is evident that F and F″ will have the same azimuth angle φ, when referred to a common zero-meridian. On the other hand the central distances of the projected poles differ thus:



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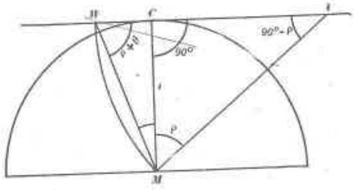

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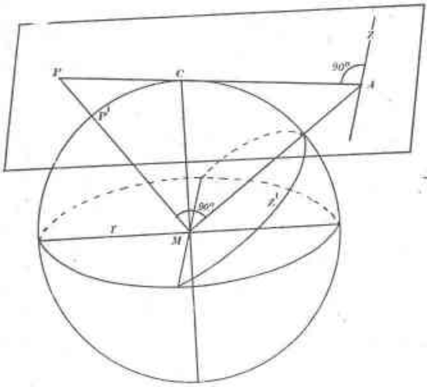

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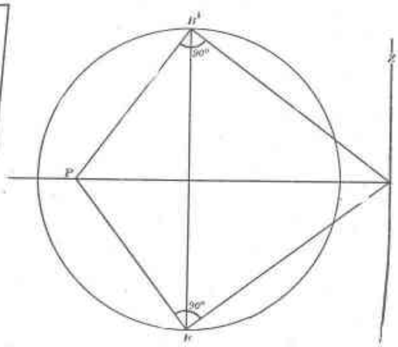

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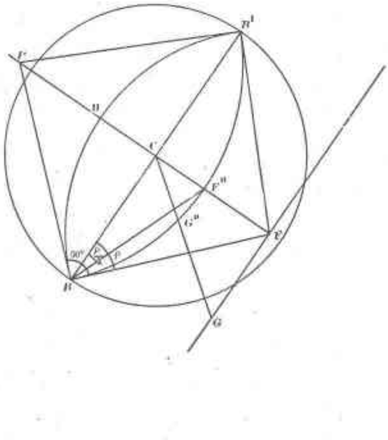

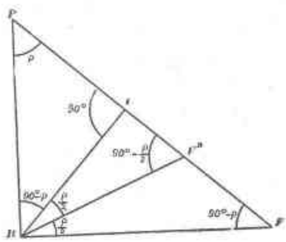

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$$\text{gnomonic:} \quad CF = r.\tan \rho$$
$$\text{stereographic:} \quad CF'' = r.\tan \tfrac{1}{2}\rho$$

We imagine the two projections superposed in the plane of the paper as in figure 15; and let F be the gnomonic face-pole, B the angle-point, of the zone FC. Then to find F″, draw BF″, bisecting the angle FBC. B and B′ will be the stereographic poles of the prism faces in zone Z (at infinity in gnomonic projection) since for a prism $\rho = 90°$ and $CB = \gamma = r.\tan 45° = r.\tan \frac{1}{2}90°$.

The stereographic projection of the zone Z will then be a circular arc thru BF″B′. The stereographic pole, G″, of any other gnomonic face-pole in Z as G is at the intersection of CG with the arc BF″B′.

The two following relations will now be proved:

1. The gnomonic edge-pole of a zone is the center of the circular arc representing the stereographic projection of the zone.

2. The angle-point of a gnomonic zone-line is the stereographic edge-pole of the zone in stereographic projection.

1. In figure 15 let P be the gnomonic edge-pole of Z. Angle $FBP = 90°$. The triangle PBF″ will be isosceles because (figure 16) Angle PBF'' = Angle $PF''B = 90° - \frac{1}{2}\rho$. Therefore $PB = PF''$. Therefore a circle with P as center and PB as radius will contain F″.

By means of this simple relation it is easy to pass from either projection to the other, zone by zone. It is more accurate to construct the stereographic projection from the gnomonic than the reverse.

2. In figure 15 let W be the angle-point of the zone Z. Then $FW = FB$. The angle at the center of the sphere subtended by CW is $\frac{1}{2}(90° - \rho)$ (see page 73); that subtended by CF″ is $\frac{1}{2}\rho$ (construction).

But WF″, regarded as a stereographically projected arc, is the sum of CW and CF″. $WF'' = \frac{1}{2}(90° - \rho) + \frac{1}{2}\rho = \frac{1}{2}90°$. WF″ is therefore the stereographic projection of a right angle and W is the edge-pole of BF″B′.

P and W, F″ and F are termed by Goldschmidt "conjugate points," and their relations are of much interest. P and W may be regarded, following the proof above, as points common to the two projections. W has the further identical property in both that it is the point at which angles between faces in the zone are measured. For the proof that the edge-pole of a stereographic zone is its angle-point, see any treatise on that projection, for example Boeke, (3) p. 14.

Equal Spacing of Zone Lines in Gnomonic Projection

Since the gnomonic projection is obtained by projection thru face-normals, the central distance of a face does not affect the position of its projection point. It however simplifies the relations of the face symbols in the projection if all faces have a common point of intersection in the plane of the projection.

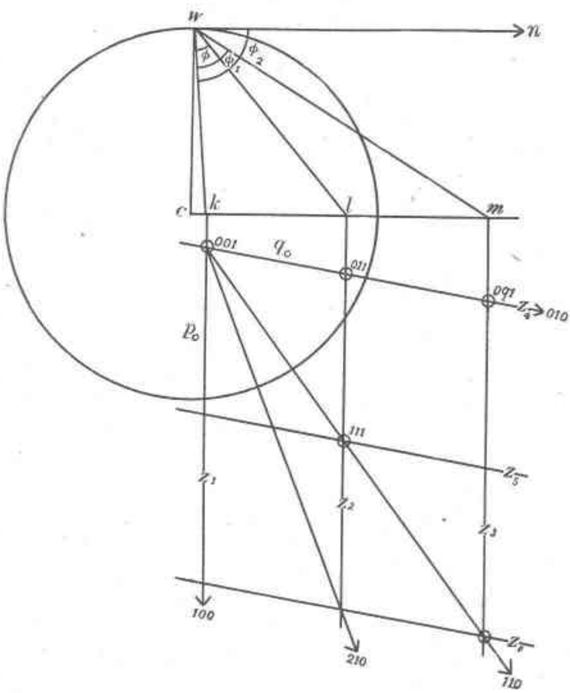

Fig. 17

Therefore conceive this plane normal to the c axis at a unit distance from the crystal center and let each face be shifted parallel to itself until it cuts that axis at unity. The Miller index of any face (hkl) will then become $\left(\frac{h}{l}\ \frac{k}{l}\ 1\right)$ which we may call (pq1). Let figure 17 be a gnomonic projection of a triclinic crystal and choose as coördinate directions the two pinacoidal zones [(001): (100)] and [(001: 010)]. Then it is true as first shown by Goldschmidt that if the unit pyramid (111) has the gnomonic coördinates p_0 and q_0, the coördinates of any face (pq1) will be pp_0 and qq_0. Since however the projection of any face-pole may be regarded as the intersection point of two zones parallel to the coördinate axes we may state the relation in the following form:

The zone lines are equidistant whenever p or q respectively increases by a like amount.

The proof involves constructions shown in the figure.

Let z_1 be the coördinate zone-line (001) to (100).

Let z_4 be the coördinate zone-line (001) to (010).

Let z_2 be the zone-line (011) to (100) determined by the coördinate q_0 of (111) measured in z_4.

Draw z_3 parallel to z_1 and z_2 so that the distance from z_2 equals the distance of z_2 from z_1.

Now the angles between the three zones are measured at the angle-point W of the common *central* to the zones, Cklm.

φ = kWl = angle (001) to (011),

φ_1 = kWm = angle (001) to (0q1),

φ_2 = kWn = angle (001) to (010).

If the face (0q1) is a crystallographically possible face then the pencil of rays at W, (k, l, m, n) must conform to the well-known condition that the anharmonic ratio of four poles in a zone must be rational (Tutton, *Crystallography*, p. 76).

That is,

$$\frac{\mathrm{km}}{\mathrm{kl}} \cdot \frac{\mathrm{nl}}{\mathrm{nm}} = \frac{\sin \varphi_1}{\sin \varphi} \cdot \frac{\sin (\varphi - \varphi_2)}{\sin (\varphi_1 - \varphi_2)} = \text{a rational quantity.}$$

Since n is at infinity, nl/nm = 1 and we have the simple relation: km/kl = a rational quantity. But km: kl = (0q1) (001) : (011) (001) = 2 : 1 (construction).

Therefore we may conclude that the zone containing 0q1 with a coördinate $2q_0$ is a possible crystal zone and we may write the symbol of the face (021). An exactly similar relation may be proved for the parallel zones z_4, z_5, z_6; and the conclusion follows that the principal zones in gnomonic projection are equally spaced in any given direction, p_0 and q_0 being the respective measures along the axial or coördinate directions. All faces in zone z_3 will have the second index 2 and similarly faces in z_6 will have the first index 2. The face at their intersection $2p_0$ $2q_0$ will therefore be (221).

This most remarkable property of the gnomonic projection gives it its greatest usefulness in graphical crystallography. A projection made from the measured angles gives at once the direction of the dominant zones as well as the constants p_0 q_0 measuring the interspaces in those zones. These constants having been fixed by the choice of the unit pyramid, the symbol of any face may be at once established by determining graphically the coördinates pp_0 and qq_0, giving the symbol for the face (pq1). For prism faces the ratio p : q or $pp_0 : qq_0$, determined by the coördinates of some face thru which the direction line of the prism passes, fixes the symbol of the form as shown in the figure.

Further, as soon as the coördinate directions and the position of any pyramid face with known indices are established, all other possible faces of the crystal may be constructed by expanding the net of parallelograms so outlined.

In later papers it is proposed to describe the characteristics of the projection in each crystal system; the methods of calculation employed; and the relations existing between the elements p_0 and q_0 and the linear elements, a, b and c.

Bibliography

The gnomonic projection has been described, to mention only the more important articles, in the following publications:—

1. Goldschmidt, V. Ueber Projection und graphische Krystallberechnung. Berlin, 1887.

   The first general discussion of the application of this projection to crystallographic investigations. Later papers in *Z. Kryst. Min.* modify and extend details.

2. Goldschmidt, V. Ueber stereographische Projection. *Z. Kryst. Min.*, **30**, 260, 1898.

   Brings out the relations of the two projections.

3. Boeke, H. E. Die gnomonische Projection. Berlin, 1913.

   An excellent and complete treatise, using Goldschmidt's methods and adding to them. Supplementary to a similar paper by the same author on the stereographic projection.

4. Smith, G. F. H. The Advantages of the Gnomonic Projection and its Use in the Drawing of Crystals. *Min. Mag.*, **13**, 309, 1903.

5. ———. The Use of the Gnomonic Projection in the Calculation of Crystals. Same Journ. **18**, 317, 1919.

Papers which deal with measurements on the three-circle goniometer and their discussion on a basis of methods very different from those of Goldschmidt.

6. Rogers, A. F. The Gnomonic Projection from a Graphical Standpoint. *Sch. Mines. Quart*, **29**, 24, 1907.

An excellent paper devoted to graphic solution in terms of polar coördinates but of the linear elements.

7. Hilton, H. The Gnomonic Net. *Min. Mag.*, **14**, 18, 1904.

A device similar to the stereographic net, useful for graphical determination of the angles between poles in gnomonic projection.

8. Hutchinson, A. A Protractor for Use in Constructing Stereographic and Gnomonic Projections of the Sphere. *Min. Mag.*, **15**, 93, 1908.

A simple wooden protractor for rapidly plotting both projections. Also historical notes on the two projections.

THAUMASITE (AND SPURRITE) FROM CRESTMORE, CALIFORNIA[1]

W. F. FOSHAG

U. S. National Museum

The contact-metamorphosed limestone at Crestmore, near Riverside, California, has yielded a large array of interesting minerals, including the species wilkeite with four acid radicles. During the summer of 1918 the mineral thaumasite, which contains three acid radicles, was found at the quarries. The first specimens discovered consisted of small masses of interlaced needles resembling the "cotton ball" ulexite. The needles were very fine and under the microscope appeared as long prisms terminated by the base or, less frequently, by a pyramid and base. The angle between the pyramid and base measured under the microscope was about 45°. Later small slender crystals and short stubby ones in parallel growth were found lining cavities and massive silky veins up to 3 cm. cutting across the contact rock. The veins closely resemble those carrying the thaumasite in Beaver Co., Utah, described by Butler and Schaller.[2]

An analysis of the mineral gave the following results:

SiO_2 9.10, $(Al, Fe)_2O_3$ 0.84, CaO 12.98, SO_3 27.56, $H_2O + CO_2$ (ign.) 49.48, sum 99.96 per cent.

The most significant feature of the occurrence is the mode of genesis of the mineral. It coats blocks of rocks thought at first to consist of monticellite, which occurs abundantly at the

[1] Published with the permission of the Secretary of the Smithsonian Institution.

[2] Am. J. Sci., [4] **31**, 131, 1911.