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RELATIONSHIP OF OPTICAL AXIAL ANGLE WITH THE THREE PRINCIPAL REFRACTIVE INDICES

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ABSTRACT

Two new charts relate the three principal refractive indices of biaxial minerals to the optic axial angle within the precision of common optical measurements. The charts solve the exact formula between these optical properties by relating partial birefringences $\beta-\alpha$ and $\gamma-\beta$ as ordinate and abscissa to 2V shown as a fan of pairs of radiating curved lines, the two lines of the pair being solutions for when β is 1.550 and 1.850.

INTRODUCTION

Measurement of optical properties of biaxial crystals gives values of the three principal refractive indices α, β and γ, the birefringence $\gamma-\alpha$ and partial birefringences $\gamma-\beta$, and $\beta-\alpha$ and the optical axial angle $2V_z$ which when correct have the well known relationships

$$\tan^2 V_z = \frac{1/\alpha^2 - 1/\beta^2}{1/\beta^2 - 1/\gamma^2} \simeq \frac{\beta - \alpha}{\gamma - \beta} \tag{1}$$

Use of this relationship thus enables the determination of concordant values of α, β and γ, the birefringence and 2V, and the detection of any discrepancies in such measurements, or in published data being used for comparison. However, direct use of this equation requires determination of the reciprocal squares of the refractive indices, and its numerical computation becomes tedious and time consuming.

Several alternative methods for using the relationship have accordingly been described. These procedures comprise computation using an approximate formula, or using a derivation of the exact formula which avoids reciprocal squares of the refractive indices; graphical solutions requiring computation of ratios of the principal refractive indices; and charts or nomograms on which the axial angle is read from the principal refractive indices, or from differences of the principal refractive indices or of their reciprocal squares.

However the graphical solutions which are most easily used only solve the approximate relationship. The new charts which are presented give a similar easily obtained solution to the exact relationship between the optical properties.

COMPUTATION METHODS

Various ways of computing the relationship between the optical constants have been described, all developed to avoid determination of the reciprocal squares of the principal refractive indices which occur in the exact equation (1).

The approximate equation given in equation (1), (Mallard, 1884) enables simple computation of $\tan^2 V_z$ as the ratio of the partial birefringences $\beta-\alpha$ and $\gamma-\beta$. Wright (1951) has recently described an equivalent approximation which avoids the square; this being

$$\cos 2V_z = \frac{(\gamma-\beta)-(\beta-\alpha)}{\gamma-\alpha} \tag{2}$$

Values of $2V_z$ obtained from α, β and γ by these approximate formulas are always higher than the correct angle. As an example, for a birefringence $\gamma-\alpha$ of 0.100 the approximate formula gives $2V_z$ up to 2° 46′ greater than the correct value when the refractive index β is 1.550, and up to 2° 20′ greater than the correct value when β is 1.850.

A better value of 2V may be obtained by suitable correction of the angle found from the approximate equations. The chart of Larsen (1921) and Larsen and Berman (1934) gives by interpolation the correction for values of $\gamma-\alpha$ to 0.300 for a refractive index α of 1.500. The correction at other values of α is obtained from this chart as that corresponding to a birefringence on the chart of $1.500/\alpha$ of the actual birefringence. An alternative method is to compute the correction to 2V. A formula for the correction to cos 2V determined by equation (2) is given by Wright (1951), who has simplified its application by tabulating the correction to 2V calculated from this formula for various values of α from 1.4 to 2.0 against various values of cos 2V.

Another equation which provides an exact relationship between 2V and α, β and γ has been described by Parker (1956). The equation determines sin V as the ratio of the sines of the angles whose cosines are the ratios α/β and β/γ. As was pointed out by Parker, this equation is convenient for solution by logarithmetic calculation.

It should be realized that the exact equation (1) given above, may also be more readily computed by using tables of reciprocal squares. Suitable tables are for example that given by Wright (1913, p. 518), or Barlow's Tables (Comrie, 1941). By means of such tables, for any specified values of α, β and γ, the exact equation (1) may be converted arithmetically to a form similar to the approximate equation.

Graphical Methods

Nevertheless rapid graphical methods of relating the optical properties of biaxial minerals are of considerable help in mineralogical and petrological studies, especially in mineral identification by optical measurements. In particular, as the principal refractive indices are commonly determined to an accuracy of about 0.002, their proper correlation with 2V requires finding not one value of 2V, but the range of values of 2V which accords with combinations of the extremes of the range to within which each principal refractive index has been determined. A number of solutions of equation (1) are thus necessary for properly correlating one set of optical data, and a graphical method or a chart to do this is a valuable help.

A variety of charts and nomograms relating the optical properties of biaxial minerals has been published. In addition a graphical method of solving the approximation, by using the ratios of the principal refractive indices in conjunction with a stereographic projection, has been described by Parker (1956); this method however has low accuracy for lower birefringences.

A number of charts relate the optical axial angle 2V, or V, to the differences in refractive indices $\beta-\alpha$, $\gamma-\beta$ or $\gamma-\alpha$. This is a very convenient relationship, as these differences are immediately related to the refractive indices, and may be one of the optical properties measured microscopically.

A small nomogram relating these values has been published by Wright (1951). In a number of other graphical solutions $\beta-\alpha$ and $\gamma-\beta$ or $\gamma-\alpha$ are scaled as abscissa and ordinate. V or 2V is then shown as a fan of lines radiating from the origin. The first of such charts was that of Wright (1911, Plate 9), which solved the approximate Mallard formula. It related $\beta-\alpha$ and $\gamma-\alpha$ with $2V_z$ for birefringences of up to 0.090.

Three charts of similar type, but relating partial birefringences $\gamma-\beta$ and $\beta-\alpha$ to V according to their exact relationship were published by Boldyrew in 1912. These charts were drawn for β of 1.500, 1.650 and 2.000, and the lines representing 2V are curved. They are however now unavailable.

Two additional charts of Wright (1913) serve to solve the exact optical relationship while retaining a fan of straight lines for V. One chart (Plate 6) has a linear scale of $(1/\beta^2-1/\gamma^2)$ and $(1/\alpha^2-1/\beta^2)$ as ordinate and abscissa, and radiating lines for V_x at 1° intervals above 10°, below which such lines would be extremely cramped. The other chart (Plate 7) uses these same functions, and differs in having the ordinate and abscissa at a scale of the square root of the difference of the reciprocal squares of the refractive indices. This produces a more open scale for smaller differences in refringence, and more uniform spacing of the lines scaling V. The necessity of determining reciprocal squares of the principal refractive indices before using these charts hinders their application to solution of the exact equation. However if the ordinate and abscissa are read as the differences $\gamma-\beta$ and $\beta-\alpha$, these charts solve the approximate Mallard formula.

Two other charts which are essentially modifications of Wright's charts have been published. Smith (1937) extended the radial lines for 2V over the whole rectangular area between ordinate and abscissa. His chart uses sliding scales on which the refractive indices are read directly, thus avoiding using the partial birefringences. More recently Gravenor (1951) has published a graphical solution to the approximate equation which is essentially the chart of Wright (1913, Plate 6) in juxtaposition to the same chart inverted. This chart is scaled for partial birefringences up to .020, at intervals of 10° in 2V. It was specifically drawn to be used in conjunction with the Hartmann net for determining the refractive index for sodium light from measurements at other wavelengths made in immersion liquids of high dispersion.

The value of the three principal refractive indices are directly related to V, according to the exact formula, by the composite grid nomogram published by Mertie (1941). On this nomogram the scale of refractive index extends from 1.45 to 2.00, and of necessity has rather small subdivisions to read to 0.002, especially for higher refractive indices, unless a very large chart is used.

The graphical solution of the exact equation published by Lane and Smith (1938) has the same disadvantage. In this chart the scales for refractive indices from 1.30 to 2.00 are only 1.6″ long at the scale of publication, and the chart requires substantial magnification to be read to an accuracy corresponding to that at which refractive indices are commonly determined.

In the chart of Waldmann (1945) the ratios α/β and γ/β are related as ordinate and abscissa to a series of lines radiating from the origin each representing a different value of 2V. Burri (1950, Plate 1) has used this type of chart, which is not suitable for smaller values of 2V, as well as requiring computation of the ratio of the refractive indices before it can be used.

Roesch and Sturenburg (1927) have also used the ratios of principal refractive indices, and give two charts which solve a modification of the exact equation for the ratios β/α and γ/α.

The New Charts

One of the most useful of the types of chart relating the optical properties of biaxial minerals is that introduced by Wright (1911) which relates the partial birefringences as ordinate and abscissa to the optic axial angle scaled as a fan of lines on the chart radiating from the origin. Single page size charts of this type can be read to the same accuracy as that of normal measurements of optical properties of minerals. However without ancillary computation such charts only solve the approximate Mallard

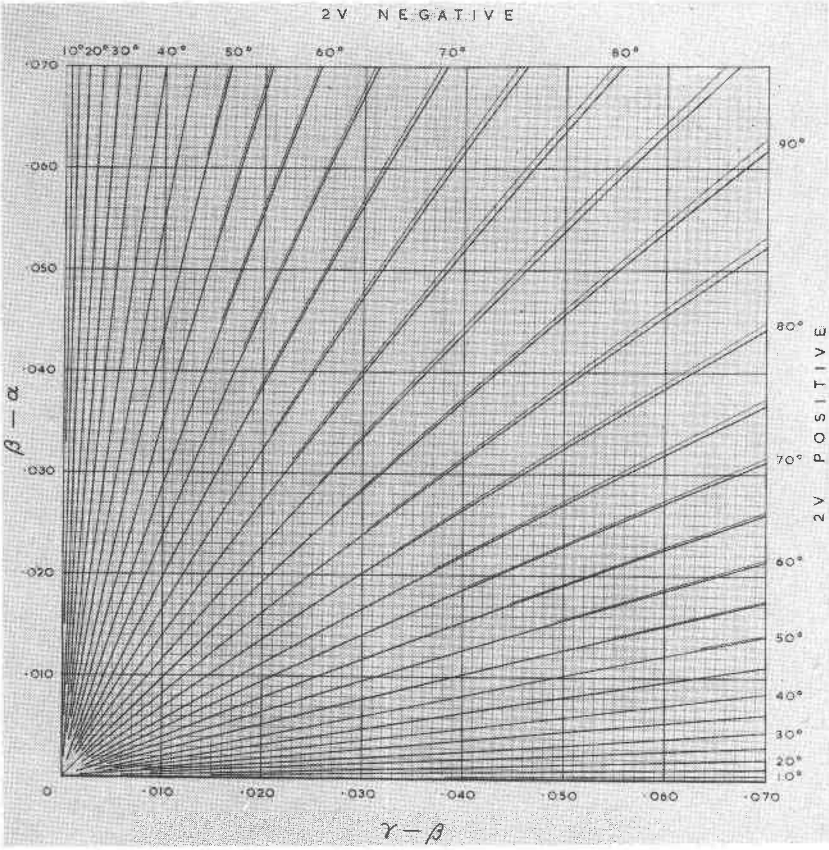

FIG. 1. Relationship of 2V to α, β and γ for $\beta=1.55$ (heavier lines) and $\beta=1.85$ (lighter lines), with linear scale of birefringence.

formula relating the optical properties, except for those of Boldyrew which are not available.

It is however not difficult to construct charts similar to these which give the exact relationship between 2V and the refractive indices, and two such charts are presented in Figs. 1 and 2. These charts compare with those of Plates 5 and 6 of Wright (1913), the ordinate and abscissa being the partial birefringences $\beta-\alpha$ and $\gamma-\beta$, while however the lines for 2V are curves plotted according to the exact equation (1) above, and are drawn throughout the chart, instead of for only half of it.

The first chart (Fig. 1) has a linear scale of partial birefringences. It is therefore read with uniform accuracy in refractive index difference throughout the chart. However such linear scaling produces marked vari-

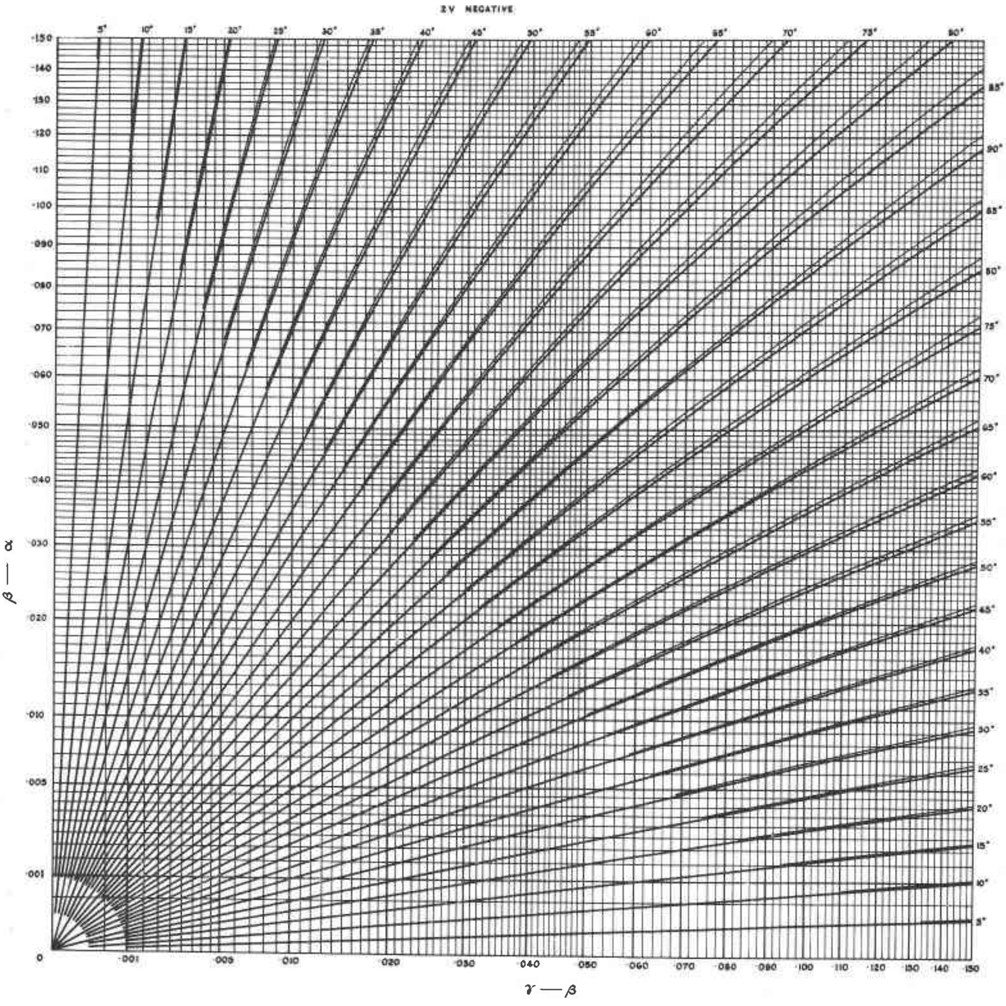

FIG. 2. Relationship of 2V to α, β and γ for $\beta=1.55$ (heavier lines) and $\beta=1.85$ (upper, lighter lines), with birefringence scaled proportional to its square root.

ation in spacing of the lines representing 2V, and considerable reduction in the accuracy for low positive and negative values of 2V. This, of course, nicely illustrates that when 2V is small, a minor variation in β, and in $\beta-\alpha$ relative to $\gamma-\beta$, produces a relatively large change in 2V.

The relationship for low values of 2V, and a low value of one partial birefringence, is more easily obtained from the second chart (Fig. 2). In this chart the partial birefringences are plotted on a scale proportional to their square root. The accuracy with which the difference in refractive index is read decreases with increasing birefringence, however the lines representing 2V are much more uniformly spaced, so that the optical relationships of biaxial minerals of low 2V are more readily determined.

The two charts give solutions of the exact equation when the refractive index β is 1.550 and 1.850. They apply for other values of β by interpolation between the pairs of curves, or by extrapolation beyond them, which can be made visually or by mensuration.

These charts were constructed by solution of the exact equation in the form

$$\frac{1}{\alpha^2} = \left(\frac{1}{\beta^2} - \frac{1}{\gamma^2}\right) \tan^2 V_z + \frac{1}{\beta^2} \tag{3}$$

which is suitable for rapid repeated computation. $1/\beta^2 - 1/\gamma^2$ and $1/\beta^2$ were determined for a number of values of $\gamma - \beta$ up to .150, with β of 1.550 and 1.850. The value of α, and hence of $\beta - \alpha$, corresponding to each of these values of $\gamma - \beta$ was then determined for each 2V curve, by machine multiplication of successively increasing values of $1/\beta^2 - 1/\gamma^2$ by $\tan^2$ V, from which $1/\alpha^2$ was recorded by addition of $1/\beta^2$. α was then read from a table of reciprocal squares and the value of $\beta - \alpha$ recorded. Points on the curves for low positive values of 2V were found by determining γ and hence $\gamma - \beta$ for selected values of α, β and V_x, using a similar procedure.

Discussion

The charts presented here clearly illustrate features of the relationship of the optic axial angle to the principal refractive indices.

The closeness of the two sets of curves for the intermediate refractive index β of 1.55 and 1.85, shows the minor effect on 2V of absolute values of the refractive indices relative to their differences, the partial birefringences. For example, for partial birefringences $\beta - \alpha$ and $\gamma - \beta$ each of 0.100, when β is 1.500 the axial angle $2V_z$ is 95° 44′. When β is increased to 2.00, the axial angle $2V_z$ is 94° 18′ for the same partial birefringences. It is clear that the difference in 2V produced by this change in β, 1° 26′, is much less than the error in 2V determined by using the approximate Mallard formula, which gives a 2V of 90° for all biaxial minerals which have equal partial birefringences. All such minerals, in which $\beta - \alpha$ equals $\gamma - \beta$, are of course optically negative. The amount their optic axial angle differs from 90° increases with increasing birefringence, while for a constant birefringence it decreases as the three principal refractive indices increase. This is shown by the progressive flattening of the curve for 2V of 90° in the chart—to fit the approximate formula this 90° line would be straight and extending from the origin exactly through the top right hand corner of the charts.

The flattening of the 2V lines away from the origin, where they correspond to the approximate formula, illustrates that $2V_z$ given by the approximate equation is always smaller than the correct angle. The approximate equation thus gives positive values of 2V which are too low, and negative values of 2V which are too high.

This is demonstrable from the formula of equation (1) above, when it is put into the form.

$$\tan^2 V_z = \frac{1/\alpha^2 - 1/\beta^2}{1/\beta^2 - 1/\gamma^2} = \frac{\gamma(1/\alpha + 1/\beta)}{\alpha(1/\beta + 1/\gamma)} \times \frac{\beta - \alpha}{\gamma - \beta} \simeq \frac{\beta - \alpha}{\gamma - \beta}$$

The coefficient $\gamma\ (1/\alpha+1/\beta)/\ \alpha\ (1/\beta+1/\gamma)$ which is omitted in the approximate formula, is always greater than one, and decreases with increasing refractive index while increasing with increasing birefringence. Omission of this coefficient to obtain the approximate formula accordingly leads to correspondingly lower values of $\tan^2 V_z$ and V_z.

These charts can be a useful aid to identification of minerals in thin sections, for having determined the birefringence $\gamma-\alpha$ and 2V, the differences $\beta-\alpha$ and $\gamma-\beta$ may be read from the charts. The three principal refractive indices α, β and γ may then be ascertained from a measurement of any one of them. Such a procedure is more simply made with Fig. 1, on which the values of $\gamma-\beta$ and $\beta-\alpha$ are found as the intersection of the 2V curve with a straight line sloping down to the right at 45°, and intersecting ordinate and abscissa at a point equal to the birefringence $\gamma-\alpha$.

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