GNOMONIC AND LINEAR HEPTAXIAL TWO-CIRCLE CALCULATION

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Abstract

Referring an hexagonal crystal to three sets of orthorhombic axes and taking $\phi_{(1-3)}$ and $\phi'_{(1-3)(=90^0-\phi)}$ alternatively, gnomonic calculation yields the following:

$\cos\phi_{1-3}$ (or ϕ'_{1-3}) $\cdot \tan\rho$

$$=\frac{k}{l}\cdot\frac{c}{a}:\frac{(h+2k)c}{l\sqrt{3a}}:\frac{(h+k)c}{la}:\frac{(2h+k)c}{l\sqrt{3a}}:\frac{h}{l}\cdot\frac{c}{a}:\frac{(h-k)c}{l\sqrt{3a}}.$$

Linear calculation yields the following:

$\tan(90°-\rho)/\cos\phi'_{(1-3)}$ (or ϕ_{1-3})

$$=\frac{l}{k}\cdot\frac{a}{c}:\frac{l}{h+2k}\cdot\frac{\sqrt{3a}}{c}:\frac{l}{h+k}\cdot\frac{a}{c}:\frac{l}{2h+k}\cdot\frac{\sqrt{3a}}{c}:\frac{l}{h}\cdot\frac{a}{c}:\frac{l}{h-k}\cdot\frac{\sqrt{3a}}{c}.$$

The linear constants are a/c and $\sqrt{3a}/c$ and the reciprocal gnomonic polar constants are c/a and $c/\sqrt{3a}$, which, as they may be interchangeable, might be indicated as p_0 and π_0. The gnomonic polar representation of the facts shown in the linear projection of a face $(hk\bar{i}l)$ is a circle with $\tan\rho$ as diameter, which is used for the graphical determination of the polar constants. The linear projection is shown to be well adapted for use in two-circle calculation.

The calculation of the reciprocal (polar) constants depends on two sets of triangles (not polar) which are homopolar, with one common angle, and with the sides adjacent to this angle having reciprocal tangents.

Five pairs of gnomonic constants are readily derived: two of which are definitely polar and refer to triaxial systems; two others are polar in the sense that they locate the projection point by co-ordinates in three and six directions respectively, and refer to tetraxial and heptaxial systems and because of the greater number of horizontal axes involved, must be fractions of the reciprocal polar units; the fifth pair are definitely auxiliary units.

Two-circle equations for the calculation of axial ratios and indices in the hexagonal system have been of two kinds; first a simple calculation dealing only with indices and axial ratios which unfortunately has attracted little attention, secondly equations involving various polar constants. In a former paper, the writer (Parsons 1937) attempted, rather unsuccessfully, to harmonize three pairs of polar constants, and in the last paragraph gave the simple equations which have led up to this paper. Later he found that Ford (1922) had given in graphical form the same information, and that Lewis (1899) had used the same principle in connection with the first order unit pyramid. For the writer, the discarding of all polar constants, except c/a and $c/\sqrt{3a}$, from the calculation, the establishment of a simple correlation of the linear and gnomonic constants, and the derivation of the curves involved in the various polar constants, has cleared up the tremendous confusion that has surrounded this system for the past sixty years.

The two-circle calculation of hexagonal constants is best accomplished by referring the crystal to three sets of orthorhombic axes with axial ratios $\sqrt{3a}:a:c$, but before entering upon the hexagonal calculation, it will be well to see what facts may be deduced from the measurement and projection of an orthorhombic crystal as well as the angles that can be determined by trigonometric calculation so that a proper selection of equations may be made for an abridged calculation of crystallographic constants.

In doing this, the writer is accepting the two-circle equations of W. H. Miller (1839, p. 83) for the orthorhombic system, which apply to the linear projection, and is making the changes that are necessary for the gnomonic projection. He is also accepting the Miller and Miller-Bravais indices, and the Miller conventions for form symbols, face symbols, zone axis symbols, and zone (circle) symbols. He would also express his admiration for the Miller precision two-circle goniometer constructed in 1874 (Lewis 1899, p. 601), which was in perfect condition in 1928 at the University of Cambridge.

The Miller (1839, p. 83) equations are as follows:

$$\tan\phi = ka/hb \qquad \phi = (hko)\wedge(100) = 90^\circ - \phi'$$
$$\tan\tfrac{1}{2}L = la/hc\cdot\cos\phi \qquad \tfrac{1}{2}L = [001]\wedge[hkl] = 90^\circ - \rho$$

In the form given, Miller's (1839, p. 79) complete two-circle calculation of the orthorhombic is perfect arithmetically, but obscure graphically. Substituting tan $90^\circ - PX$, etc. for cot PX, etc., the equations are identical with those used in the linear calculation in this paper. Substituting $1/\tan PX$, etc., for cot PX, etc., the equations, when inverted, are identical with those used for the gnomonic calculation. Miller's equations have been used by nearly all crystallographers in the past century, sometimes in a mutilated condition. They have never been surpassed; when changed, it has always been for the worse.

Conventions in Notation

Following Miller's practice, the form symbol is given as $\{hk\bar{\imath}l\}$, the face symbol as $(hk\bar{\imath}l)$; the zone axis as $[\overline{uvu+vw}]$, and the zone circle symbol as $[hk\bar{\imath}l,\ h0\bar{h}0]$. Miller used $[\overline{uvu+vw}]$ interchangeably for the zone axis and the zone circle except in one paragraph (p. 48) but commonly used the zone symbol as given above for the zone circle only. This usage enables one to use six types of angles without ambiguity, as follows: $(010)\wedge(110)$, the well known interfacial angle, $[001]\wedge(001)$ or $[001]\wedge(hkl)$ giving ρ_0 or ρ, $(001)\wedge[hkl,\ 100]$, not used in this paper, $[001]$

$\wedge$[001, 010] and [001]$\wedge$[*hkl*, 010], giving ξ_0 and $\xi_0+\xi$, [001]$\wedge$[*hkl*], useful in the linear calculation, [001, 010]$\wedge$[*hkl*, 010], giving ξ.

Preliminary Orthorhombic Projection and Calculation

The projection of a beryl crystal (Fig. 1) will be used to illustrate orthorhombic calculation both in linear and gnomonic projection. The rectangular grid in the front half gives multiples of $c/\sqrt{3a}$ and c/a so that the orthorhombic symbol of $(21\bar{3}1)$ may be taken as (511). The line $A'F'$ in the rear half is the linear projection of the same face with the origin below the projection. The angle of azimuth on the a_2 axis (hexagonal) is indicated as ϕ_2 and its complement as ϕ_2', as later similar angles will be referred to the a_1 and a_3 axes.

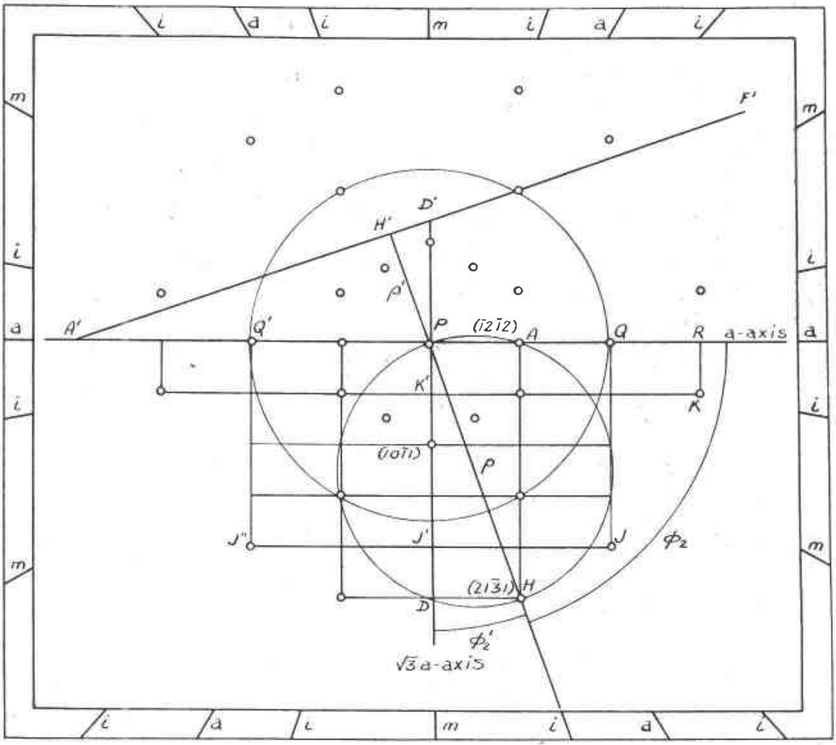

Fig. 1

Now considering the orthorhombic as a special case of the triclinic, we will see in Table 1 the information that we have in connection with (511), $(21\bar{3}1)$, together with the remaining angles to be solved.

TABLE 1. ANGLES FOUND IN TWO-CIRCLE MEASUREMENT OF A CRYSTAL FOR THE FACE (511) ORTHORHOMBIC, $(21\bar{3}1)$ HEXAGONAL

Orthorhombic	Hexagonal
$\alpha = b \wedge c = [001] \wedge [010] = 90°$	$[0001] \wedge [10\bar{1}0]$
$\beta = a \wedge c = [100] \wedge [001] = 90°$	$[\bar{1}2\bar{1}0] \wedge [0001]$
$\gamma = a \wedge b = [100] \wedge [010] = 90°$	$[\bar{1}2\bar{1}0] \wedge [10\bar{1}0]$
$\lambda = (001) \wedge (010) = 90°$	$(0001) \wedge (\bar{1}2\bar{1}0)$
$\mu = (001) \wedge (100) = 90°$	$(0001) \wedge (10\bar{1}0)$
$\nu = (010) \wedge (100) = 90°$	$(\bar{1}2\bar{1}0) \wedge (10\bar{1}0)$
$\rho_0 = [001] \wedge [001] = 0°$	$[0001] \wedge (0001)$
$\rho = [001] \wedge (511) =$	$[0001] \wedge (21\bar{3}1)$
$\rho' = [001] \wedge [511] = 90° - \rho$	$[0001] \wedge [21\bar{3}1]$
$\phi_0 =$ ———	
$\phi = (510) \wedge (010)$	$(21\bar{3}0) \wedge (\bar{1}2\bar{1}0)$
$\phi' = (510) \wedge (100)$	$(21\bar{3}0) \wedge 10\bar{1}0)$
$\eta_0 = [001] \wedge [001, 100] = 0°$	$[0001] \wedge [0001, 10\bar{1}0]$
$\xi_0 = [001] \wedge [001, 010] = 0°$	$[0001] \wedge [0001, \bar{1}2\bar{1}0]$
$\eta_0' = [001] \wedge [010] = 90°$	$[0001] \wedge [\bar{1}2\bar{1}0]$
$\xi_0' = [001] \wedge [100] = 90°$	$[0001] \wedge [10\bar{1}0]$
To find in the gnomonic projection	
$\eta = [001, 100] \wedge [511, 100]$	$[10\bar{1}0, 0001] \wedge [21\bar{3}1, 10\bar{1}0]$
$\xi = [001, 010] \wedge [511, 010]$	$[\bar{1}2\bar{1}0, 0001] \wedge [21\bar{3}1, \bar{1}2\bar{1}0]$
$a, b, c.\ h, k, l$	$\sqrt{3}a, a, c.\ h, k, i, 2h+k, h+2k\ h-k, .l$
And in the linear projection	
$\eta' = [001, 100] \wedge [011] = 90° - \eta$	$[10\bar{1}0, 0001] \wedge [\bar{1}2\bar{1}2]$
$\xi' = [001, 010] \wedge [501] = 90° - \xi$	$[\bar{1}2\bar{1}0, 0001] \wedge [50\bar{5}1]$

The calculation of these angles follows·

$\cos\phi \cdot \tan\rho = \tan\ \eta = k/l \cdot c/a$ (IV and (III)

$\cos\phi' \cdot \tan\rho = \tan\ \xi = h/l \cdot c/\sqrt{3}a$ (IV), $= (2h+k)/l \cdot c/\sqrt{3}a$ (III)

$\tan\rho'/\cos\phi = \tan\eta' = l/k \cdot a/c$ (IV and (III)

$\tan\rho'/\cos\phi' = \tan\xi' = l/h \cdot \sqrt{3}a/c$ (IV) $= l/(2h+k) \cdot \sqrt{3}a/c$ (III)

AXIAL RELATIONS

Before going into the graphical and mathematical solution of an hexagonal problem, it will be well to consider the axial relations and intercepts of a plane on the hexagonal axes. It has been customary to refer an hexagonal crystal to three horizontal axes (a_1, a_2, a_3) of unit length and a vertical axis (c) which is greater or less than unity, and the recorded ratios are $a:c$. It has, however, been recognized that there is a second set of horizontal axes, at right angles to the a axes, having a length of $\sqrt{3}a$, which could be used if results obtained from the first set proved unsatisfactory. This seems to indicate that the complete solution of an hexagonal problem demands that the relationship of six horizontal axes to the vertical axis should be ascertained.

In x-ray analysis of hexagonal crystals, it is customary to look upon the hexagonal cells as being made up of three unit cells, each an orthorhombic prism with angles of 60° and 120°.

The results of measurement of an hexagonal crystal when plotted in a gnomonic projection indicate three units with axes $\sqrt{3}a:a:c$. In this way we get six horizontal axes, three pairs of rectangular axes, which yield the simplest polar equation of the plane $(hk\bar{i}l)$ for each of the triaxial units involved.

Constants in the Linear Projection

In the linear projection (Fig. 2), in order to have a common centre for the three orthorhombic units, they are shown as interpenetrating, and the height of the vertical axis is indicated by a circle with radius equal to

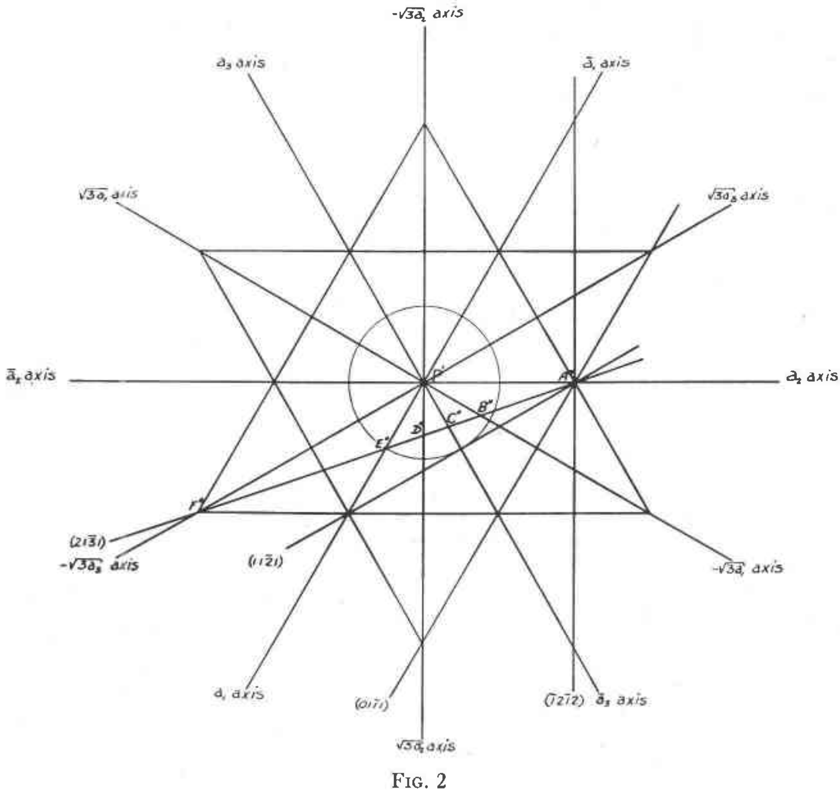

Fig. 2

c (beryl). Of the four faces shown, we will for the present confine our attention to the general case $(21\bar{3}1)$, which cuts the a_2 axis at unity. Careful measurement will show that:

$$\frac{P'A''}{c}:\frac{P'B''}{c}:\frac{P'C''}{r}:\frac{P'D''}{c}:\frac{P'E''}{c}:\frac{P'F''}{c}$$

$$=\frac{l}{1}\cdot\frac{a}{c}:\frac{l}{4}\cdot\frac{\sqrt{3}a}{c}:\frac{l}{3}\cdot\frac{a}{c}:\frac{l}{5}\cdot\frac{\sqrt{3}a}{c}:\frac{l}{2}\cdot\frac{a}{c}:\frac{l}{1}\cdot\frac{\sqrt{3}a}{c}$$

$$=\frac{l}{k}\cdot\frac{a}{c}:\frac{l}{h+2k}\cdot\frac{\sqrt{3}a}{c}:\frac{l}{h+k}\cdot\frac{a}{c}:\frac{l}{2h+k}\cdot\frac{\sqrt{3}a}{c}:\frac{l}{h}\cdot\frac{a}{c}:\frac{l}{h-k}\cdot\frac{\sqrt{3}a}{c}$$

which are the abscissae cut off on six axes by the trace of the plane $(21\bar{3}1)$ in the linear projection with the origin *above* the plane of projection and with $r_0=c$. The linear projection of the same face is shown in Fig. 3 with the origin *below* the plane of projection and with $r_0=1$.The ratios shown above are the tangents of interaxial angles (zonal axes).

Constants in the Gnomonic Projection

If we invert the terms of the equations under linear constants so as to have reciprocal values, we get

$$\frac{c}{P'A''}:\frac{c}{P'B''}:\frac{c}{P'C''}:\frac{c}{P'D''}:\frac{c}{P'E''}:\frac{c}{P'F''}$$

$$=\frac{k}{l}\cdot\frac{c}{a}:\frac{h+2k}{l}\cdot\frac{c}{\sqrt{3}a}:\frac{h+k}{l}\cdot\frac{c}{a}:\frac{2h+k}{l}\cdot\frac{c}{\sqrt{3}a}:\frac{h}{l}\cdot\frac{c}{a}:\frac{h-k}{l}\cdot\frac{c}{\sqrt{3}a}$$

$$=\frac{PA}{r_0}:\frac{PB}{r_0}:\frac{PC}{r_0}:\frac{PD}{r_0}:\frac{PE}{r_0}:\frac{PF}{r_0}\quad\text{(Fig. 3)}$$

which are the indices with the axial ratios, and the abscissae, respectively, on the axes of the gnomonic projection when a circle is described with $\tan\rho$ as the diameter. They are also the tangents of interzonal angles.

Linear Zonal Axes and Gnomonic Zone Lines

The line joining c and A'' (Fig. 2) is common to $(21\bar{3}1)$, $(11\bar{2}1)$, $(01\bar{1}1)$, and $(\bar{1}2\bar{1}2)$ and is the axis $[\bar{1}2\bar{1}2]$ of the zone $[\bar{1}2\bar{1}2, 10\bar{1}0]$ in which these four faces lie. In like manner cB'', cC'', cD'', cE'', and cF'' are zone axes which are perpendicular to the planes which intersect in the normal to the plane $(21\bar{3}1)$ (Fig. 3). The traces of these planes in the gnomonic projection are the familiar zone lines in the projection. The angle between one of these zonal planes and r_0 on the a axes is known as η, and to distinguish the particular axis is indicated in this paper as η_1, η_2, and η_3. The corresponding angle on the $\sqrt{3}a$ axes is known as ξ, with similar indication of the axis involved. The angles subtended between the zonal axes and the perpendicular in the linear projection are $90°-\eta$, and

$90° - \xi$, respectively. These zone axes are best indicated in terms of a possible face as follows: $cA''[\bar{1}2\bar{1}2]$, $cB''[02\bar{2}1]$, $cC''[33\bar{6}2]$, $cD''[50\bar{5}2]$, $cE''[2\bar{1}\bar{1}1]$, and $cF''[1\bar{1}02]$. In the transfer of the linear projection of $(21\bar{3}1)$ (Fig. 2) to the gnomonic projection (Fig. 3) there is a reversal of the sign for l/h, l/k, and $l/h+k$.

Hexagonal Indices

In the hexagonal system, it is customary to use four indices, h, k, $\bar{i}$, and l, with three of these referring to horizontal units and one to the vertical unit. Of the horizontal values, one (i) equals the sum of the other two (h and k) but with opposite sign ($\pm$). Of the twenty-four ways in which these letters may be combined, four at least are given in standard works on mineralogy and crystallography. In a previous paper, the writer used the order given by Williams (1901). In this paper the usage of Dana is followed, as indicated above, but inasmuch as indices on six horizontal axes are to be considered, the following values will be sought:

$$\frac{h}{l}, \ \frac{k}{l}, \ \frac{(h+k)}{l}, \ \frac{(h+2k)}{l}, \ \frac{(2h+k)}{l}, \ \frac{(h-k)}{l}.$$

Although the familiar transformation formula from the standard to the alternative orientation is involved, no further reference to this will be made, but in the gnomonic projection (Fig. 3), these values will be shown as multiples of c/a and $c/\sqrt{3}a$, together with their reciprocals in the linear projection which are directly connected in Fig. 4 with the intercepts on the axes in Fig. 2.

Gnomonic Calculation of c/a and $c/\sqrt{3}a$

In the accompanying projection (Fig. 3), we have the gnomonic projection of the forms $c\{0001\}$, $o\{11\bar{2}2\}$, $p\{10\bar{1}1\}$, $s\{11\bar{2}1\}$, $v\{21\bar{3}1\}$, $m\{10\bar{1}0\}$, $a\{11\bar{2}0\}$, and $i\{21\bar{3}0\}$ of beryl with zone lines parallel to the a axes in the front half. In the rear half is shown the linear projection of the face $(21\bar{3}1)$ $A'F'$, passing through the origin 5 cm. below the plane of projection. The graphical solution will at once be clear to those who know that $\tan \rho\ (\bar{1}2\bar{1}2) = c/a$ and $\tan \rho (10\bar{1}2) = c/\sqrt{3}a$.

Dropping perpendiculars from H, $(21\bar{3}1)$, to each of the six axes, we have $PA = c/a$, $PB = 4c/\sqrt{3}a$, $PC = 3c/a$, $PD = 5c/\sqrt{3}a$, $PE = 2c/a$, and $PF = c/\sqrt{3}a$. This is an extension of Ford's (1922) graphical determination of h, k, and i.

For the complete mathematical calculation, six angles of azimuth are used: ϕ_1, $(21\bar{3}0) \wedge (2\bar{1}\bar{1}0)$, $\phi_1' = 90° - \phi_1$, ϕ_2, $(21\bar{3}0) \wedge (\bar{1}2\bar{1}0)$, $\phi_2' = 90° - \phi_2$, ϕ_3, $(21\bar{3}0) \wedge (11\bar{2}0)$ and $\phi_3' = 90° - \phi_3$.

The cosine of each of the ϕ angles is multiplied by tan ρ to obtain tan η, the tangent of the angle of slope of the three zones involved, and the cosine of each of the ϕ' angles is used similarly to obtain tan ξ, the angle of slope of the other three zones involved with axial ratios, e.g. $\cos\phi_2 . \tan\rho = \tan\eta_2$, the angle of slope of [$21\bar{3}1$, $10\bar{1}0$], whose zone axis is

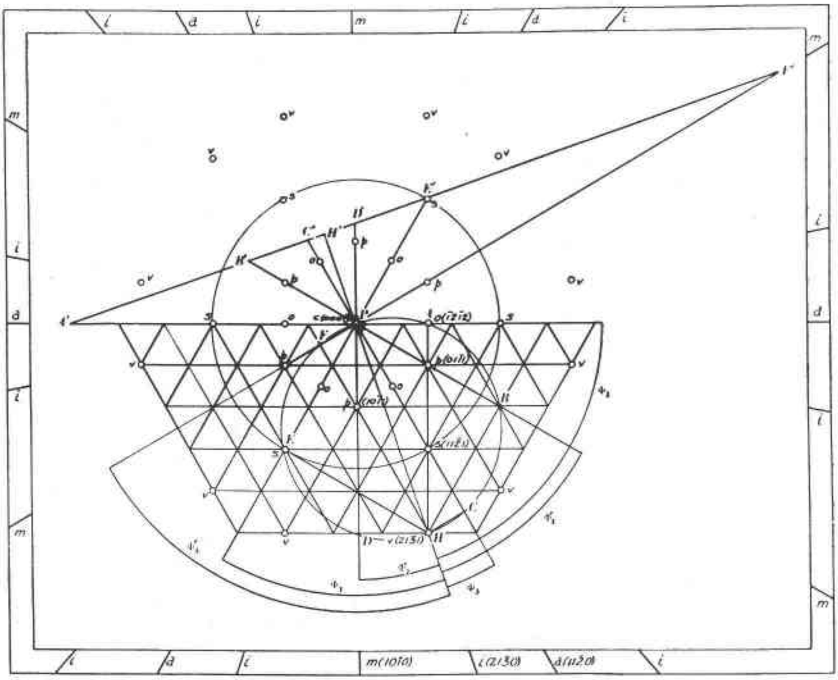

FIG. 3

[$\bar{1}2\bar{1}2$]. This form for the equation involved was apparently first used by Lewis (1899, pp. 428 and 449) for the pyramid ($01\bar{1}1$) but was not extended to the general case.

This fundamental calculation is the one from which all other gnomonic constants are derived by multiplication or division by a constant, and is the only one that gives the polar constants as reciprocals of the intercepts on the linear axes. The complete calculation for ($22\bar{3}1$) follows:

$$\cos\phi_2 \cdot \tan\rho = PA/PH \cdot PH/r_0 = \tan\eta_2 = k/l \cdot c/a \quad = l/PA' = r_0/PA' \quad (1)$$

$$\cos\phi_1' \cdot \tan\rho = PB/PH \cdot PH/r_0 = \tan\xi_1 = (h+2k)/l \cdot c/\sqrt{3}a = l/PB' = r_0/PB' \quad (2)$$

$$\cos\phi_3 \cdot \tan\rho = PC/PH \cdot PH/r_0 = \tan\eta_3 = \frac{(h+k)}{l} \cdot c/a \quad = l/PC' = r_0/PC' \quad (3)$$

$$\cos\phi_2' \cdot \tan\rho = PD/PH \cdot PH/r_0 = \tan\xi_2 = \frac{(2h+k)}{l} \cdot c/\sqrt{3}a = l/PD' = r_0/PD' \quad (4)$$

$$\cos\phi_1 \cdot \tan\rho = PE/PH \cdot PH/r_0 = \tan\eta_1 = h/l \cdot c/a = l/PE' = r_0/PE' \quad (5)$$

$$\cos\phi_3' \cdot \tan\rho = PF/PH \cdot PH/r_0 = \tan\xi_3 = \frac{(h-k)}{l} \cdot c/\sqrt{3}a = l/PF' = r_0/PF'. \quad (6)$$

These final results are shown graphically in a single plane in Fig. 4, where

$$r_0/PA' : r_0/PE' : r_0/PC' = c/P'A'' : c/P'E'' : c/P'C''$$

and

$$P'A'' : P'E'' : P'C'' \text{ (Fig. 4)} = P'A'' : P'E'' : P'C'' \text{ (Fig. 2)}$$

and

$$r_0/PF' : r_0/PB' : r_0/PD' : = c/P'F'' : c/PB'' : c/P'D''$$

and

$$P'F'' : P'B'' : P'D'' \text{ (Fig. 4)} = P'F'' : P'B'' = P'D'' \text{ (Fig. 2)}$$

Linear Calculation of a/c and $3a/c$

The line $A'F'$ (Fig. 3) is the linear projection of $(21\bar{3}1)$ with the origin below the plane of projection ($r_0 = 1$). The calculation of the elements concerned follows:

$$\tan(90° - \rho)/\cos\phi_2 = (PH'/r_0)/(PH'/PA') = \tan\eta_2' = \frac{l}{k} \cdot \frac{a}{c} = PA'/r_0$$

$$\tan(90° - \rho)/\cos\phi_1' = (PH'/r_0)/(PH'/PB') = \tan\xi_1' = \frac{l}{h+2k} \cdot \frac{\sqrt{3}a}{c} = PB'/r_0$$

$$\tan(90° - \rho)/\cos\phi_3 = (PH'/r_0)/(PH'/PC') = \tan\eta_3' = \frac{l}{h+k} \cdot \frac{a}{c} = PC'/r_0$$

$$\tan(90° - \rho)/\cos\phi_2' = (PH'/r_0)/(PH'/PD') = \tan\xi_2' = \frac{l}{2h+k} \cdot \frac{\sqrt{3}a}{c} = PD'/r_0$$

$$\tan(90° - \rho)/\cos\text{-}\phi_1 = (PH'/r_0)/(PH'/PE') = \tan\eta_1' = \frac{l}{h} \cdot \frac{a}{c} = PE'/r_0$$

$$\tan(90° - \rho)/\cos\phi_3' = (PH'/r_0)/(PH'/PF') = \tan\xi_3' = \frac{l}{h-k} \cdot \frac{\sqrt{3}a}{c} = PF'/r_0.$$

The final results are in every case reciprocal to the results of the gnomonic calculation and establish definitely that the linear constants are a/c and $\sqrt{3}a/c$ and that the reciprocal polar constants of the gnomonic projection are c/a and $c/\sqrt{3}a$.

With the introduction of the circle with radius of r_0, the linear projection assumes importance in helping to solve crystallographic problems and is worthy of further investigation, particularly in the inclined systems.

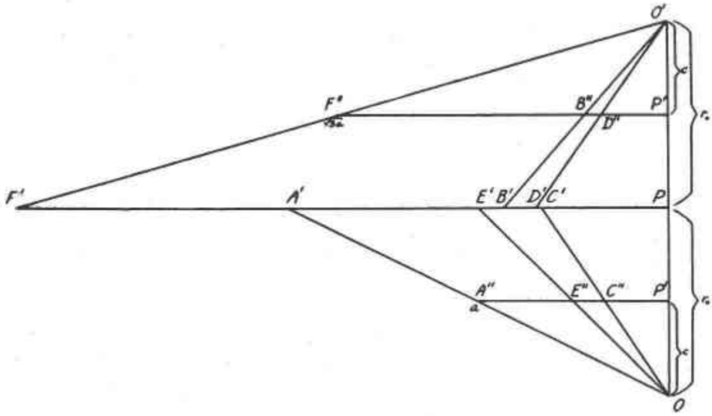

FIG. 4

For Prisms

The calculation of the indices for prismatic faces is as follows:

$$\cos\phi_1 : \cos\phi_2 : \cos\phi_3 : = h : k : i.$$

Abridgements to the Calculation

It is well to let the student see at least one complete calculation which will show definitely the relationship between indices and parameters on all seven axes, and then abridgements to the calculation may well be introduced, depending altogether on what is desired. Equations 1, 3, and 5, give the complete calculation for the crystal when referred to the four standard axes of reference. If only the best average values of c/a and $c/\sqrt{3}a$ are desired, equations 3 and 4 should be sufficient. If indices are desired for forms where l is large or for forms which cannot be readily be shown on the projection, equations 1 and 5 give $k/l \cdot c/a$ and $h/l \cdot c/a$, from which all other desired indices can be derived.

The student will be perfectly satisfied with such an abridgement, whereas he is never satisfied with the explanation that one of the indices has been lost in the projection but can be obtained by adding the two others.

Equation for tan ρ $(hk\bar{i}l)$

From equations (1) and (4), we find

$$\tan\rho(hk\bar{i}l) = \sqrt{\frac{k^2c^2}{l^2a^2} + \frac{(4h^2 + 4hk + k^2)}{l^2} \cdot \frac{c^2}{3a^2}} \qquad (7)$$

and from equations (2) and (5)

$$\tan \rho(hk\bar{i}l) = \sqrt{\frac{(h^2 + 4hk + 4k^2)}{l^2} \cdot \frac{c^2}{3a^2} + \frac{h^2c^2}{l^2a^2}} \tag{8}$$

and from equations (3) and (6)

$$\tan \rho(hk\bar{i}l) = \sqrt{\frac{(h^2 + 2hk + k^2)}{l^2} \cdot \frac{c^2}{a^2} + \frac{(h^2 - 2hk + k^2)}{l^2} \cdot \frac{c^2}{3a^2}}. \tag{9}$$

Each of these equations when reduced to a common denominator and simplified, gives

$$\tan \rho(hkil) = \sqrt{\frac{4(h^2 + hk + k^2)}{l^2} \cdot \frac{c^2}{3a^2}} \tag{10}$$

$$= \frac{2\sqrt{h^2 + hk + k^2}}{l} \cdot \frac{c}{\sqrt{3}a}$$

which is the equation of tan ρ when referred to three sets of orthorhombic axes with ratios $a{:}\sqrt{3}a{:}c$. It is simultaneously the equation for a circle having tan ρ as its radius and the equation for the diameter ($=\tan \rho$) of a circle in terms of its supplementary chords. The first of these is extremely useful in analyzing the results of measurement in terms of $2c$ and $2c/\sqrt{3}$, while the second gives the polar elements c and $c/\sqrt{3}$ directly.

Graphical Determination of Polar Constants from the Equation for Tan ρ

In the gnomonic projection (Fig. 3), with $PH = \tan \rho(21\bar{3}1)$ as diameter, describe the circle $PABC\ DEF$ cutting the six axes at A, B, C, D, E, and F. Then $PA = c/a$, $PB = 4c/\sqrt{3}a$, $PC = 3c/a$, $PD = 5c/\sqrt{3}a$, $PE = 2c/a$, and $PF = c/\sqrt{3}a$.

The polar units derived in this manner are c/a and $c/\sqrt{3}a$: they are the polar units of a hexagonal crystal treated as a special case of the orthorhombic system.

Orientation of the Projection

In the hexagonal system there are two types, one giving a triangular pattern in the gnomonic projection, the other an hexagonal pattern. For the first of these, the rhombohedral, it is only necessary to draw the principal zone lines which are parallel to the a axes, having the apex of the inner triangle $(10\bar{1}1)$ to the front.

The second is more complicated, but, in general, the zone lines with the greatest number of projection points are perpendicular to the a axes. If there are only first order pyramids, or second order pyramids, or dihexagonal pyramids, the orientation cannot be established beyond question. Any two of these ordinarily can establish the orientation beyond reasonable doubt, so as to give the simplest indices for all the forms.

An interesting exception to both these statements is exhibited in a crystal of hematite described by Foshag (1920) which, except for triangular markings on the base, exhibits perfect hexagonal symmetry and shows $(10\bar{1}1)$ and $(01\bar{1}1)$ truncating the edges of $(22\bar{4}3)$.

Polar and Auxiliary Constants

The calculation of the axial ratios and polar constants c/a and $c/\sqrt{3}a$ brings us to the common point where all theories of axial relations in the hexagonal system meet. If we go no further, it leaves us in the position of considering the hexagonal system as a special case of the orthorhombic system. Graphically, these constants have been determined as units of measure of chords of a circle. Three other circles give three pairs of constants, two of which are definitely connected with theories of axial relations while the third seems to be more general in scope. In addition, zone lines drawn perpendicular to any pair of a axes or $\sqrt{3}a$ axes give grids in terms of the polar constants $p_0(G_1)$ or $p_0(G_2)$.

The four pairs of polar constants are as follows:

(1) Hexagonal heptaxial polar constants $c/3a$ and $c/3\sqrt{3}a$ derived by describing a circle with diameter equal to tan $\rho/3$ thus cutting the six axes at one-third the distance obtained for the orthorhombic constants. The six abscissae taken in any order and moved parallel with themselves, when necessary, locate the projection point of the face normal by co-ordinates in six directions.

(2) Hexagonal tetraxial polar constants p_0 and Π_0 (Goldschmidt 1886), derived by describing a circle with diameter equal to 2 tan $\rho/3$, thus cutting the six axes at two thirds the distance obtained for the orthorhombic constants. Alternate abscissae give two sets of three which locate the projection point by co-ordinates in three directions as shown by the writer (1938) for p_0, although as a matter of fact, $p_0(G_2)$, which is arithmetically equal, was used. Goldschmidt could have obtained these constants only from the polar equation of a plane derived from the equation of the plane in terms of the intercepts on four hexagonal axes. This was verified independently for the writer many years ago by Dean Samuel Beatty, Professor of Mathematics in the University of Toronto, but until the simple method described above was found, the writer could see no way of deriving these constants with their proper indices by graphical methods.

(3) Orthorhombic polar constants c/a and $c/\sqrt{3}a$. These have already been shown.

(4) The polar constants $p_0(G_1)$ and $p_0(G_2)$ (Goldschmidt 1886) best derived by drawing zone lines perpendicular to any pair of a axes for $p_0(G_1)$ and any pair of $\sqrt{3}a$ axes for $p_0(G_2)$. These locate the projection

point of the face normal by co-ordinates on *two* inclined axes.

An extremely useful pair of auxiliary constants, $2c/a$ and $2c/\sqrt{3}a$, is obtained from abscissae on the zone lines, which intersect in the projection point of $(hk\bar{i}l)$, when cut by a circle with radius equal to $\tan\rho$. When $r_0=5$ cm., we get direct measurement of c/a and $c/\sqrt{3}a$ as units of measurement of interfacial spacing in the zones shown in the gnomonic projection.

It will be noted that only one pair of the four pairs of polar constants is referred to alternative sets of three horizontal axes and belongs to the hexagonal system, as ordinarily presented. These and the heptaxial constants are polar only in the sense that they locate the projection point of the plane $(hk\bar{i}l)$ by co-ordinates in three of six directions, respectively, in the gnomonic projection. In every other respect, they must be looked upon as auxiliary constants. When more than two horizontal axes are involved, the projection units for locating the projection point of $(hk\bar{i}l)$ will be fractions of the normal polar units.

Summary

The reciprocal relations of linear and polar constants in the hexagonal system are shown graphically in linear and gnomonic projections and the complete mathematical calculation from ϕ and ρ angles is given for each projection, with suggested abridgements for ordinary use. The calculations are based on the conception that an hexagonal crystal should be referred to three sets of orthorhombic axes, seven axes in all. Polar constants which have been proposed in the past introduce unnecessary sources of error and should be discarded as a means of calculating axial ratios.

$$\tan\rho = \frac{2\sqrt{h^2+hk+k^2}}{l}\cdot\frac{c}{\sqrt{3}a}.$$

The polar constants c/a and $c/\sqrt{3}a$ with their proper indices are determined graphically by chords of a circle with $\tan\rho$ as diameter, which in the gnomonic projection is the reciprocal of the trace of the given plane in the linear projection. Two other pairs of polar constants are derived by drawing circles with diameter of $\tan\rho/3$ and $2\tan\rho/3$ and a fourth pair by drawing zone lines perpendicular to any pair of a axes or $\sqrt{3}a$ axes. The linear projection with the addition of a circle with radius $r_0=1$ is shown to be well adapted to two-circle calculations in systems which can be referred to rectangular axes, and when referred to the same origin as the gnomonic projection gives reciprocal relations which are easily recognized. Zonal axes in the linear projection are located by the intersection of the traces of planes having a common horizontal intercept;

zone lines in the gnomonic projection, which are parallel to the $\sqrt{3}a$ axes, connect the projection points of faces having a common h/l, k/l, or i/l and furnish the best means of orientating the projection, the lines with the greatest number of faces being parallel with the a axes in the rhombohedral group and perpendicular to the a axes in crystals with six-fold symmetry.

In conclusion, the writer would pay high tribute to Miller, Goldschmidt, G. F. H. Smith, Lewis, Palache, and Ford, for their contributions to two-circle goniometry. The only changes introduced in their fundamental equations, which are arithmetically correct, involve, in some cases for graphical clarity, the substitution of $1/\tan \rho$ or $\tan (90° - \rho)$ for $\cot \rho$; $\cos (90° - \phi)$ for $\sin \phi$; and $1/\cos (90° - \phi)$ for $\operatorname{cosec} \phi$.

He would also thank Dr. E. H. Kraus, of the University of Michigan, Dr. W. A. Wooster and Dr. F. Coles Phillips, of the University of Cambridge for references establishing the date of the original Miller two-circle goniometer, which preceded the instrument seen by the writer. Finally, he would thank Dr. E. W. Nuffield, of the University of Toronto for making the drawings which illustrate this paper.

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