# SHIFT OF THE PLANE OF PROJECTION IN THE GNOMONIC PROJECTION

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Abstract

In the gnomonic projection the shift from one plane of projection to another can be accomplished graphically by several different methods; by their use the observer is able to locate the positions of diametral planes and of single points in the new projection. Several new methods, based on the isometric lines and their polar planes, are described briefly, together with the existing methods. The usefulness of the gnomonic projection as an aid in photogrammetric mapping from airplane photographs is emphasized.

The gnomonic projection is widely used in crystallography, navigation, and other branches of science, for the reason that all great circles on the sphere appear as straight lines in the projection. The plane of projection is the plane tangent to the sphere at a given point called the pole of the projection. In many problems it is desirable to change from one plane of projection to another. This can be done with the aid of a special gnomonic projection plot, or by graphical methods, or by computation. The projection plot, however, is at a disadvantage because, for polar angles exceeding 70°, the plot is so large that it becomes unwieldy. Recourse must then be had to some method of graphical construction or to computation. Several of the graphical methods are known,[1] but others described below appear to be novel and may merit brief description, although they are based on known principles.

In Fig. 1a let the plane of the paper be the plane of projection, tangent at $P$ to the sphere of reference of radius $PC$. The actual center of the sphere, $O$, is directly beneath $P$. The straight line, $FNS$, represents a great circle in projection and is, in fact, the line of intersection of the diametral plane, whose trace on the sphere is the given great circle, with the tangent plane of projection. The plane of projection is now shifted, so that another direction, $OP_1$ represented by the point $P_1$ in the present projection, becomes the pole. The problem is to find the position of the great circle plane, $FNS$, and of any given point, such as $B$, in the projection after the rotation. The problem can be solved graphically with the aid of certain construction lines and points indicated in Figs. 1a to 1d.

[1] V. Goldschmidt, Ueber Projektion und Graphische Krystallberechnung, pp. 67–71, **1887**, *Berlin*.

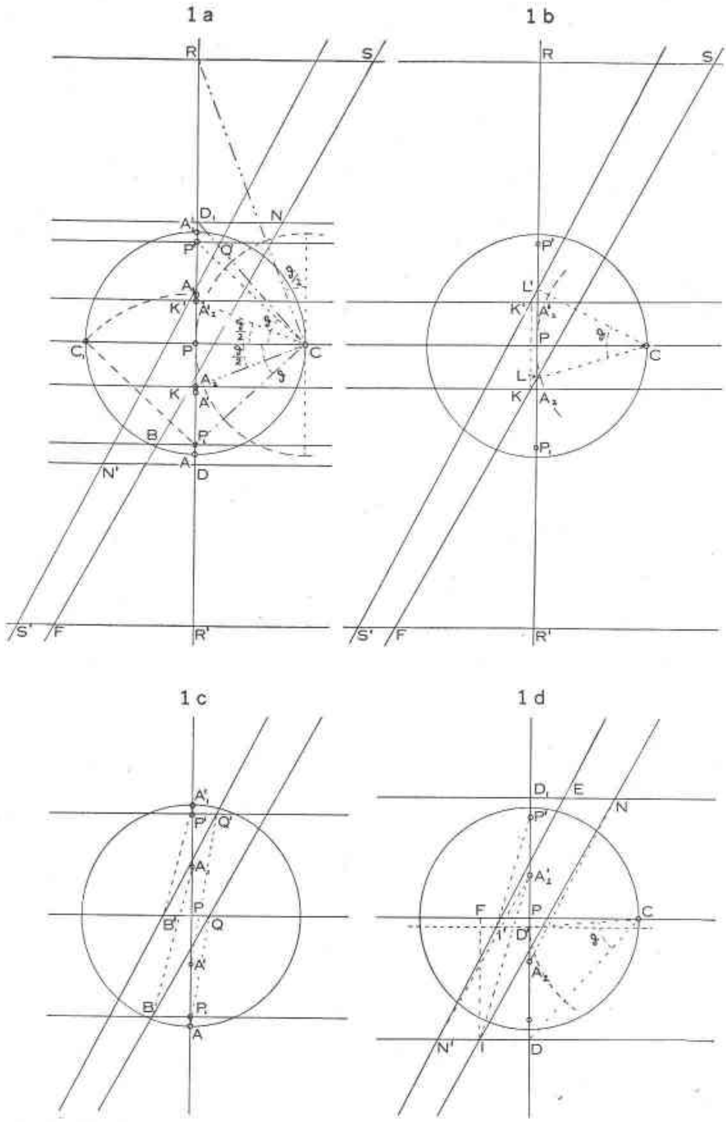


FIG. 1a. In this figure the several constructional lines and points are shown which may serve as aids in determining the positions of zone lines and points in the gnomonic projection after its rotation through a given angle $\varphi$.

FIG. 1b. Illustrates the methods for locating the points, $L'$ on the principal line, $K'$ on the isometric line, and $S'$ on the isometric polar line.

FIG. 1c. Indicates how the points, $B'$ on the normal line, and $Q'$ on the new guide line are found.

FIG. 1d. Illustrates the methods for ascertaining the positions of the points, $N'$ on the new prism line, $I'$ on the new prism line after rotation, also the direction of the desired line $N'E$ parallel with the line $A_2N$.

The shift of the projection from pole $P$ to pole $P_1$ is essentially a rotation of the projection about the horizontal axis parallel to $PC$ (Fig. 1a), and through the center, $O$, of the sphere. During this rotation, points in the vertical plane normal to the axis and represented by the *principal line* $P_1PP'$ remain in the plane; also all great circle planes containing the axis of rotation remain perpendicular to the line $P_1PP'$ and their traces in the projection are parallel lines. To measure the angular rotation, rotate the vertical plane $P_1PP'$ about the line $P_1P'$ as an axis and bring the center of the sphere, $O$, into the plane of projection at $C$. The actual angle of rotation from $P_1$ to $P$ is then the angle $PCP_1=\theta$. On rotation $P_1$ becomes the new pole and $P$, the original pole, is shifted to $P'$. The original equatorial plane, of which $P$ is the pole, becomes the line $N'D$ in the new projection; in like manner the line $D_1N$, which is the great circle plane normal to $P_1$, ($P_1CD_1=90°$), becomes the equator after the rotation. The line $A_2K$ is midway, in angular measure, between $P_1$ and $P$; angle $PCA_2$=angle $P_1CA_2=\theta/2$. On rotation, $A_2$ is shifted to $A_2'$ and the line $A_2K$ to $A_2'K'$. The line $A_2K$ is important because it represents the line of intersection of the new and old planes of projection; it is common to both projections; similarly the line $A_2'K'$. The points on $A_2K$ retain the same relative positions on $A_2'K'$; thus the point $K$ is shifted to $K'$ and $A_2'K'=A_2K$. In stereophotogrammetry these lines are the *perspective* lines, so named because on them the scales are similar and not distorted.

To find the angle between any two points, as $P_1$ and $B$ (Fig. 1a), in a given great circle plane bring the center, $O$, of the sphere to $A_1$ in the plane of projection by rotating the plane about the line $P_1B$. The distance $P_1O$ to the center of the sphere is equal to $P_1C=P_1C_1$. Draw $P_1A_1=P_1C_1$ and find at $A_1$ the desired position of the center of the sphere brought into the plane of projection. The point $A_1$ is called the *angle point* of the great circle plane $P_1B$. The angle $P_1A_1B$ is the angle between the radial lines represented by the points $P_1$ and $B$. Similarly $A_1'$, $A'$, $A_2$ and $A_2'$ are the angle points of the great circle planes $PC$, $P'Q'$, $D_1N$, and $DN'$ respectively.

As possible constructional aids we have then the angle point $C$ from which the angle between any two points on the *principal line* $P'P_1$ can be read off directly; the *normal line* $PC$ with its angle point $A$; the old and new *guide lines* $P_1B$ and $P'Q'$ with the angle points $A_1$ and $A'$; the old and new *prism* or *polar guide lines*,

$D_1N$ and $DN'$ with the angle points $A_2$ and $A_2'$; the *common* or *perspective lines* $A_2K$ and $A_2'K'$; the *common polar planes* $RS$ and $R'S'$, polar respectively to $A_2$ and $A_2'$. With the aid of these constructional elements the observer may choose any two of the following operations to ascertain the position of any given great circle plane or zone line, such as $FNS$, in the new projection.

(1) The point $K$, the intersection of the given line with the original isometric line, becomes $K'$ in the new projection. (Fig. 1b). $A_2K = A_2'K'$.

(2) The intercept $R'S'$ on the polar plane to $A_2'$ is equal to the intercept $RS$ on the $RS$ line polar to $A_2'$. (Fig. 1b). Angles $A_2CR = A_2'CR' = 90°$ (Fig. 1a).

(3) The point $L$ at the intersection of $FS$ with the principal line $PP'$ (Fig. 1b) is shifted to $L'$; the angle $LCL'$ is equal to the angle of rotation, $\theta$, of the projection.

(4) The angle between $P$ and $Q$ (Fig. 1c) on the normal line is $PAQ$. Draw through $A'$, the angle point of $P'Q'$, a line parallel with $AQ$ and find $Q'$.

(5) The angle between $P_1$ and $B$ is $P_1A_1B$. Through $A_1'$ draw $A_1'B'$ parallel with $A_1B$ and locate the point $B'$. (Fig. 1c).

(6) Draw through $A_2'$ a line parallel with $IN$ and locate the point $N'$. (Fig. 1d). The desired line $N'E$ is parallel with the line $A_2N$. (Fig. 1d).

(7) The plane $ID$ is shifted to $I'D'$ by rotation of the projection. (Angle $DCD' = \theta$. Fig. 1d). Draw through $P'$, the angle point of $I'D'$, a line parallel with $IA_2'$ and thus locate $I'$. (Fig. 1d).

The seven points obtained by these different operations are all located on the desired straight line which represents the position of the old line $FNS$ in the new projection. Experience has shown that the position of $FNS$ in the original projection determines, in a measure, the selection of the operations to use in ascertaining the position of the line after rotation of the projection. With the exception of isometric lines and their polar planes the methods are based on the use of angle points of definite construction lines.

To find the position of any given point, $I$, (Fig. 1d) after rotation of the plane of projection, method 7 is probably the simplest. In the original projection the given point is at the intersection of two coordinate lines, $ID$ and $IF$. The angle point of $ID$ is at $A_2'$; after rotation, the line $ID$ is at $I'D'$ whose angle point is $P'$. The angle between the plane $IA_2'$ and the principal plane, $DP'$, is $IA_2'D$.

Draw through $P'$, the angle point of $D'I'$, a line parallel with $IA_2'$ and locate the point $I'$ on $I'D'$. Angle $I'P'D'$ is then equal to angle $IA_2'D$. Therefore $I'$ is the position of the given point after the rotation.

In crystallography a crystal drawing is made by projecting in parallel perspective the zone axes on a given plane of projection. Thus, if a top view or plan is desired, perpendiculars are erected to the different zone lines in gnomonic projection and the crystal faces are represented as they appear projected orthographically on the equatorial plane. Any other plane of projection is obtained by drawing its trace in the projection, by locating its angle point, and then, by erecting a line perpendicular to the line passing through the angle point and the point of intersection of a zone line with the given trace, the direction of the edge between any two faces in the given zone is obtained. The same directions can be obtained by first rotating the projection so that the desired plane of projection is the equatorial plane and then erecting the normals to the zone lines in their new positions. For planes of projection not too far from the equatorial plane of the original projection, this method is serviceable. It was the first method used for preparing crystal drawings from the gnomonic projection[2] and is analogous to the usual method adopted for preparing a crystal drawing from a stereographic projection. But ordinarily the plane of projection chosen for crystal drawings is near the pole of the original projection and the crystal faces in the central part of the original gnomonic projection are so far away from the pole in the new projection that the standard procedure is preferable, either with or without the aid of a preliminary top view or plan drawing.

In gnomonic charts for great circle sailing the foregoing methods for shifting from one pole of projection to another are useful in preliminary work. The degree of accuracy obtainable is not great because the earth's geoid surface is not strictly spherical.

In phototopographic work the gnomonic projection enters to a certain extent. The photograph of a plane area taken from an airplane is a gnomonic projection of that area on the plane of the negative. If the area is covered with a rectangular network or grid for measuring purposes, an oblique view of the grid shows the lines converging toward definite points on the horizon. The view is, in

[2] V. Goldschmidt, Ueber Projektion und Graphische Krystallberechnung, pp. 82–85, **1887**, *Berlin*.

fact, a gnomonic projection whose plane includes with the original plane an angle equal to the angle between the tilted negative and the ground. By rotating the original gnomonic projection, on which the grid lines are rectangular and equally spaced, through the angle of tilt the new positions of the lines, as they appear on the photographic negative, are obtained.

In mapping flat areas from airplane photographs the use of the gnomonic projection is an aid to the observer in visualizing the spatial relations involved. The *perspective center* is the center, $O$, of the projection sphere; the *principal distance* is the radius of the projection sphere. The *principal point* is the pole of the projection; the *principal line* is the vertical $P_1PP'$ of Fig. 1a; the *perspective planes* are great circle planes. The *horizon trace* is the prism or polar guide line $D_1N$ of Fig. 1a. The *horizontal plane* is the equatorial plane of the projection. On the perspective lines the scales are similar and not distorted. The *angle of tilt* is the angle of rotation, $\theta$, of the gnomonic projection plane. Having given the angle of tilt, it is a simple matter for the observer to deduce the directions of grid or other lines in an oblique photograph; and vice versa, to ascertain from an oblique photograph their directions and positions on the horizontal plane of mapping and thus to prepare the map with the aid of the gnomonic projection.