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A TABULATION OF THE 32 CRYSTAL CLASSES[1]

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The 32 crystal classes, each characterized by its unique combination of symmetry elements, are now firmly established. Representatives of all but one of them (the trigonal dipyramidal class) have been found, either among minerals or laboratory products. The prediction of Hessel,[2] professor of mineralogy at the University of Marburg, of the thirty-two crystal classes in 1830, when only seventeen of them were known, must rank as one of the notable achievements of science in the nineteenth century.

Since there are not many textbooks that include a complete list of the 32 crystal classes, the writer presents the accompanying tabulation. While it must be recognized that in an elementary course in mineralogy not more than ten or eleven crystal classes (classes 2, 5, 8, 15, 18, 19, 20, 27, 30, 31, and 32 are the most important) can be studied in any detail, there are convincing arguments in favor of presenting a complete list of the classes for the consideration of the student. In some respects the table of crystal classes bears the same relation to geometrical crystallography that the periodic table of elements does to inorganic chemistry.

In the tabulation here presented there are seven vertical columns which give in order: (1) the crystal system, (2) the number of the class, (3) the name of the class, (4) the number of faces in the general form of the class, (5) symbols for the symmetry of the class, (6) a prominent example of the class, and (7) the symbols used by Schoenflies.

1. CRYSTAL SYSTEMS

Six systems are recognized instead of seven, the hexagonal being divided into two subsystems, the rhombohedral and the

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[2] *Krystallometrie, oder Krystallonomie und Krystallographie,* **1830,** (reprinted in *Ostwald's Klassiker der Exakten Wissenschaften,* Nos. 88 and 89, *Leipzig,* **1897**).

hexagonal. This is the plan adopted by Fedorov, who, however uses syngony and hyposyngony instead of system and subsystem.

One important point that should be emphasized here is that the trigonal dipyramidal (no. 21) class and the ditrigonal dipyramidal (no. 22) class belong to the hexagonal subsystem, or to the hexagonal system proper if one prefers seven systems, and not to the trigonal or rhombohedral subsystem (or system). In six books published within the last decade, the two classes mentioned have been assigned to the trigonal system or to the trigonal division of the hexagonal system. Evans and Davies,[3] who correctly assign these two classes to the hexagonal system proper, state that Hilton in 1907 was the first to point out that this is their proper place but they are in error here, for Bravais[4] as early as 1849 placed these two classes in the hexagonal (sénaire) system proper and not in the trigonal or rhombohedral (ternaire) system.

That the trigonal dipyramidal and ditrigonal dipyramidal classes belong to the hexagonal subsystem or hexagonal system proper and not to the trigonal or rhombohedral subsystem or system is definitely proved by the application of the theory of groups.[5] The trigonal dipyramidal (no. 21) and ditrigonal dipyramidal (no. 22) classes are subgroups of the dihexagonal dipyramidal (no. 27) class, but not of the hexagonal or ditrigonal scalenohedral (no. 20) class.

A still better argument for placing these two classes in the hexagonal subsystem or hexagonal system proper is found in the fact that these classes (nos. 21 and 22) together with classes nos. 23 to 27 inclusive have as their space lattice a hexagonal prism only, while classes 16 to 20 inclusive have either a rhombohedron or hexagonal prism as a space lattice.[6] We know from X-ray work of the last decade or so that the space lattices of crystals are fundamental.

There is general agreement on the assignment of the other classes to the systems indicated.

[3] Elementary Crystallography, p. 130, *London*, (1924).

[4] *Jour. de l'Ecole Polytechnique.* Vol. 34. Reprinted in *Etudes Cristallographiques*, III ^{me} partie, pp. 273–274, *Paris* (1866).

[5] See paper by the writer, *Proc. Am. Acad. Arts and Sci.*, vol. **61**, p. 199 (1926).

[6] Hilton, *Mathematical Crystallography*, pp. 239–240, *Oxford* (1903).

2. Numbers of the Classes

The classes are numbered, the class without any symmetry being no. 1 and the class with the highest grade of symmetry being no. 32. This seems to be a better arrangement than the one that places the hexoctahedral class first and the asymmetric class last. It does not mean that the student necessarily begins his crystallography course with the triclinic system. The writer in presenting the subject of elementary crystallography always starts with the orthorhombic system. There are at least two textbooks[7] in which the orthorhombic system is treated first.

The order of the classes in the tabulation is the same as Groth's arrangement,[8] except that the trigonal dipyramidal class is placed immediately after the hexagonal scalenohedral class. Groth's numbers 19, 20 and 21 then become, respectively, my numbers 21, 19 and 20. When this change is made, the order of the classes is only in minor part arbitrary. Each of the 32 classes constitutes a mathematical "group"[8a] with its "subgroups" and as has been shown recently by the writer,[9] with the present arrangement of the classes the subgroup of each group appears before the group itself. This is not true of Groth's arrangement.

Even if classes 16 to 20 inclusive are treated as a separate rhombohedral or trigonal system they should not be separated from classes 21 to 27 inclusive.

The present arrangement of the classes first appeared in a textbook by the writer.[10]

3. Names of the Classes

The name of the class given in the third column is an adjective derived from the name of the general form except that in the case of the class without any symmetry the term asymmetric is preferred to pediad,[11] a name derived from the one-faced form,

[7] Rogers, *Introduction to the Study of Minerals*, 1st ed. *New York*, **1912**, Evans and Davies, *loc. cit.* (1924).

[8] *Physikalische Krystallographie*, 4th ed., pp. 335-337, *Leipzig* **(1905)**.

[8a] See footnote 15, p. 575.

[9] *Proc. Am. Acad. Arts and Sci.*, vol. **61**, p. 200 (1926).

[10] *Introduction to the Study of Minerals and Rocks*, 2nd ed., p. 80, *New York*, **1921**.

[11] The term used by Lewis, *A Treatise on Crystallography*, p. 148, *Cambridge* **(1899)**, who introduced it. Most authors use "pedial" instead of "pediad."

pedion. This method, which was first used by Groth,[12] is in my opinion the best way of designating the various crystal classes. Groth's names for the isometric classes may be simplified in accordance with the suggestions of Moses.[13] It should be noted that the term dipyramidal is used instead of bipyramidal. Wulff was the first to make this change, but he has been followed by Groth and others[14] in recent years.

It has been objected that some of these names for the classes are cumbersome. It is true that more than half of them are rather long, but the point may be emphasized that the student must become familiar with the names of the general forms in any event. If the same names are used for the crystal classes then he is not required to learn a new set of names, which is a distinct advantage.

Names referring to the varying types of merohedrism (hemihedrism, hemimorphism, tetartohedrism, and ogdohedrism) are also widely used. The use of terms involving merohedrism imply that crystal systems are fundamental whereas we know now that crystal classes and not crystal systems are fundamental. In view of this fact, ideas based upon merohedrism are at present of historical interest only.

In referring to a particular system it is convenient to use a general term for the class with the maximum symmetry in that system. For this purpose many authors use "holohedral," but "holosymmetric" is preferable for the reason given in the preceding paragraph.

Other sets of names for the various crystal classes have been proposed by Dana, Lewis, Miers, Spencer, Evans, Wherry, and others.

While there may be slight advantages in some of the class names proposed by these authors, the writer has come to the conclusion that names of crystal classes based upon names of general forms furnish the most consistent logical scheme of nomenclature that has yet been devised. Groth's names are more widely used than any other set of names; they are logical, based upon a uniform plan, are free from ambiguity, and are international in character.

[12] *loc. cit.*

[13] *Characters of Crystals*, pp. 57–58, *New York*, (1899).

[14] See Boldirev, *Zeit. f. Kryst. u. Min.*, vol. **62**, pp. 145–150 (1925).

4. Number of Faces in the General Form

The numbers, 1, 2, 3, 4, 8, 12, 16, 24, and 48, of the fourth column are the number of faces in the general form of each class. Each class constitutes a group.[15] There are as many operations in the group as there are faces in the general form and the number of operations defines "the order of the group." These numbers have been called "symmetry numbers" by Shearer.

5. Symbols for Symmetry Elements

In the fifth column of the table symbols for the symmetry elements are given. The combined symbols for any class are sometimes known as the "symmetry formula."

A_n stands for an n-fold axis of symmetry, P, for a plane of symmetry, and C, for a center of symmetry. The symbols P and C were used by Bravais, and A_n was first used, as far as can be ascertained, by Voigt.[16] A number before a symmetry element refers to the number of axes or planes of symmetry present. The compound symbol, $\overline{AP}_n$ used for an axis-plane of rotatory-reflection was devised by the writer.[17] The compound symbol, $\overline{CA}_n$, also devised by the writer,[18] is used for an axis-center of rotatory-inversion.

In a recent mathematical study of crystal symmetry the writer[19] in order to account for all the faces of the general forms of crystals found it necessary to employ both rotatory-reflections and rotatory-inversions in addition to ordinary rotations, reflections, and inversion.

[15] "Group" is a mathematical term used for a series of operations which have the following properties: (1) "The product (the result of one operation followed by another) of any two of them is equivalent to another operation of the series." (2) "The inverse of any operation is also a member of the series." The symmetry operations of a crystal (rotation, reflection, inversion, rotatory-reflection, and rotatory-inversion) form a group since the above conditions are fulfilled.

[16] *Die Fundamentalen Physikalischen Eigenschaften der Krystalle*, pp. 191–193, *Leipzig* (1898).

[17] *Introduction to the Study of Minerals*, p. 7, *N. Y.* (1912). The logotype symbol AP, but with the right limb of the letter A coincident with the vertical line of P, is now widely used as an abbreviation for "Associated Press," but the writer claims priority!

In this article it is necessary to substitute the symbol $\overline{\mathrm{AP}}$ for the logotype symbol.

[18] *Proc. Am. Acad. Arts and Sci.*, vol. **61,** p. 171 (**1926**).

[19] *Proc. Am. Acad. Arts and Sci.*, vol. **61,** pp. 161–203(**1926**).

Symmetry elements include or imply "powers" of operations, that is, some operations may be obtained by repeating another operation a certain number of times. For example, $a_{180°}$ is the second "power" of $a_{90°}$ and $a_{270°}$ the third "power." The following tabulation shows the relation between symmetry elements and symmetry operations. Here and elsewhere in this article lower case letters refer to operations and upper case letters to elements. 1 is the symbol for identity or the identical operation.[20]

Symmetry Elements	Symmetry Operations
A_2	$=1, a_{180°}$
A_3	$=1, a_{120°}, a_{240°}$
A_4	$=1, a_{90°}, a_{180°}, a_{270°}$
A_6	$=1, a_{60°}, a_{120°}, a_{180°}, a_{240°}, a_{300°}$
P	$=1, p$
C	$=1, c$
$\overline{AP_4}$	$=1, \overline{ap}_{90°}, a_{180°}, \overline{ap}_{270°}=\overline{CA_4}$
$\overline{AP_6}$	$=1, \overline{ap}_{60°}, a_{120°}, c, a_{240°}, \overline{ap}_{300°}$
$\overline{CA_6}$	$=1, \overline{ca}_{60°}, a_{120°}, p, a_{240°}, \overline{ca}_{300°}$

When $n/2$ is even, rotatory-reflections and rotatory-inversions are equivalent; in this case rotatory-reflections are to be preferred. With these operations and only with these may all the faces of the general forms of the 32 classes be accounted for.

There remains to be explained that some of the symbols in the symmetry column are enclosed in parentheses and some in brackets. Parentheses indicate that an inversion or a reflection is one of the powers, respectively, of a rotatory-reflection or rotatory-inversion. Brackets indicate that the symmetry element so designated has one or more operations in common with the symmetry element that precedes it. For example, in classes 25 and 27 $a_{120°}$ and $a_{240°}$ are common to A_6, $\overline{AP}_6$, and $\overline{CA}_6$, and so the two latter symbols are enclosed in brackets.

It is recognized that axes or planes of symmetry designated by nA or nP are not always equivalent, but to indicate non-equivalent axes or planes by different exponents would often make the symmetry symbols too complicated. Class 15, for example would become $A'_4\ [\overline{AP}'_4]\cdot 2A''_2\cdot 2A'''_2\cdot P'\cdot 2P''\cdot 2P'''\cdot C$.

[20] This is a device for mathematical completeness. For an n-fold axis of symmetry, there are n operations, provided the identical operation is used.

6. Examples

The sixth column of the table gives an example of each of the crystal classes except class 21, for which no example has yet been found. Examples of minerals are given by name and examples of prepared compounds of the laboratory are indicated by the chemical formula.

7. Symbols of Schoenflies

The last column gives the symbols used by Schoenflies.[21] These point group symbols are used for the most part as a basis for designating space groups. For example D_{3d}^2 is one of the space groups of the point group $D_3{}^d$. Some of the objections to the symbols of Schoenflies have been pointed out by Wyckoff.[22] Wyckoff emphasizes the desirability of adopting a more simple and logical notation of symbols for point groups and space groups and suggests a set of symbols which is a modification of a set put forward by Hilton. I agree with Wyckoff that we need a simple, convenient, and logical notation of point group symbols (more condensed than symmetry symbols), but in view of the necessary use of both rotatory-reflections and rotatory-inversions as symmetry elements, it is not advisable to use such expressions, for example, as 3Ci and 3Di as point group symbols for the rhombohedral and hexagonal scalenohedral classes, respectively. The same objection may be raised to the symbols of Hilton[23] and Evans.[24]

Use of Group Theory in Geometrical Crystallography

In conclusion I wish to call attention to the advantage of using the mathematical theory of groups in the advanced study of geometrical crystallography.

An analytical study of crystal symmetry with the aid of the theory of groups establishes definitely just what constitutes the symmetry operations and symmetry elements of crystals.[25]

The use of operations introduces into crystallography a dynamic conception of geometry which becomes most fascinating.

[21] *Theorie der Krystallstruktur*, Leipzig, (1923).

[22] *Am. Jour. Sci.*, [5], vol. **6**, pp. 288–290 (1923).

[23] *Mineralog. Mag.*, vol. **14**, p. 261 (1907); *ibid.*, vol. **19**, p. 319 (1922).

[24] *Nature*, vol. **113**, pp. 80–81, (1924).

[25] See paper by the writer, *The Mathematical Study of Crystal Symmetry*, *Proc. Am. Acad. Arts and Sci.*, vol. **61**, pp. 161–201 (1926).