

The theoretical composition based on this assumption is given in Table 2, Column 2.

TABLE 2. THEORETICAL COMPOSITION OF CANCRINITE

	1	2
SiO ₂	32.4	33.6
Al ₂ O ₃	27.5	31.0
CaO	7.9	8.7
Na ₂ O	16.72	13.7
CO ₂	5.4	5.8
H ₂ O	5.4	4.4

Neither of these interpretations are entirely satisfactory. So far it has not been possible to attribute the variations to albite in solid solution as has been shown to be the cause of the variations in nephelite. It seems quite certain, however, that nephelite or carbonate-nephelite will be found to be an essential constituent.

THE ADDITION AND SUBTRACTION RULE IN GEOMETRICAL CRYSTALLOGRAPHY¹

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The addition or subtraction of the Miller symbols of two faces of a zone taken index by index gives another face of the zone. That is to say faces with the symbols $(h+p.k+q.l+r)$ and $(h-p.k-q.l-r)$ are in the zone $[hkl:pqr]$. This useful fact, now to be proved, may for convenience be known as the *addition and subtraction rule*.

PROOF OF THE ADDITION AND SUBTRACTION RULE

The intercept equations of planes through the origin parallel to any two faces, (hkl) and (pqr) , are, in terms of the intercepts a, b, c of a unit face, (111) , respectively:

$$\frac{hx}{a} + \frac{ky}{b} + \frac{lz}{c} = 0 \dots\dots\dots (1)$$

$$\frac{px}{a} + \frac{qy}{b} + \frac{rz}{c} = 0 \dots\dots\dots (2)$$

¹ Paper presented at the annual meeting of the Mineralogical Society of America at Ithaca, N. Y., December 31, 1924, under the title, AN INTERESTING AND USEFUL PROPERTY OF ZONES. See abstract in *Am. Mineral.*, 10, 68 (1925).

Solving each equation for z we have the following equation:

$$c \left(-\frac{hx}{la} - \frac{ky}{lb} \right) = c \left(-\frac{px}{ra} - \frac{qy}{rb} \right)$$

Dividing by c and collecting we have:

$$\frac{px}{ra} - \frac{hx}{la} = \frac{ky}{lb} - \frac{qy}{rb}$$

Multiplying by lr we obtain:

$$\frac{x}{a}(lp - hr) = \frac{y}{b}(kr - lq).$$

If both sides of this equation are divided by $(lp - hr)(kr - lq)$, the result is:

$$\frac{x}{a(kr - lq)} = \frac{y}{b(lp - hr)} \dots \dots \dots (3)$$

Solving equations (1) and (2) for y in a similar manner the following result is obtained:

$$b \left(-\frac{hx}{ak} - \frac{lz}{ck} \right) = b \left(-\frac{px}{aq} - \frac{rz}{cq} \right).$$

Dividing by b and collecting we have:

$$\frac{hx}{ka} - \frac{px}{qa} = \frac{rz}{qc} - \frac{lz}{kc}$$

Multiplying by kq we obtain:

$$\frac{x}{a}(hq - kp) = \frac{z}{c}(kr - lq)$$

Dividing by $(hq - kp)(kr - lq)$ we have:

$$\frac{x}{a(kr - lq)} = \frac{z}{c(hq - kp)} \dots \dots \dots (4)$$

Combining equations (3) and (4) we have:

$$\frac{x}{a(kr - lq)} = \frac{y}{b(lp - hr)} = \frac{z}{c(hq - kp)} \dots (5)$$

This is the same form as $x/x_1 = y/y_1 = z/z_1$ which is the symmetrical equation of a right line from the origin through the point

with the coordinates, $x = x_1$, $y = y_1$, and $z = z_1$. Equation (5), then, represents a line from the origin to the point with the coordinates, $x = a(kr - lq)$, $y = b(lp - hr)$, and $z = c(hq - kp)$. This line, which is known as a zone-axis, is a line through the origin parallel to the intersection line of the (hkl) and (pqr) faces, since equations (1) and (2) are satisfied by the same values for x , y , and z . The zone $[hkl:pqr]$, then, may be represented by the expression $[kr - lq, lp - hr, hq - kp]$ which is called the zone symbol. The symbol is usually expressed as $[uvw]$.

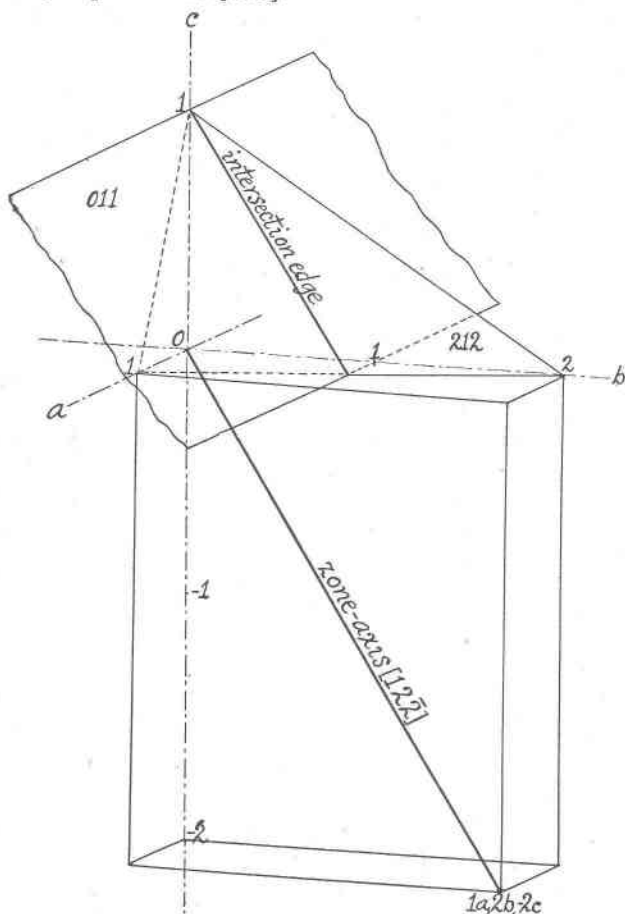


Fig. 1.
The intersection edge and zone-axis of two faces (011) and (212) , of an orthorhombic crystal.

The relations indicated are made clearer by Fig. 1. Here the faces (011) and (212) of an orthorhombic crystal intersect in a line parallel to the zone-axis which is a line from the origin to a point with the coordinates $x = (kr - lq)a$ or $1a$, $y = (lp - hr)b$ or $2b$, and $z = (hq - kp)c$ or $-2c$. The zone symbol is $[12\bar{2}]$.

The empirical rule of obtaining zone-indices by cross-multiplying face-indices, thus:

$$\begin{array}{c|ccc|c} h & k & l & h & k & l \\ & \times & \times & \times & & \\ p & q & r & p & q & r \end{array}$$

is due to the fact that the indices of the zone $[hkl:pqr]$ are $[kr - lq, lp - hr, hq - kp]$.

Let (efg) be any face in the zone $[hkl:pqr]$. The equation of a face through the origin parallel to efg is

$$\frac{ex}{a} + \frac{fy}{b} + \frac{gz}{c} = 0 \dots \dots \dots (6)$$

where a , b , and c are the intercepts of the unit face (111). If in equation (6) we substitute for x/a , y/b , and z/c the values of these obtained from equation (5) and expressed in terms of one of them (y/b) we have the following:

$$\frac{ey(kr - lq)}{b(lp - hr)} + \frac{fy}{b} + \frac{qy}{b} \frac{(hq - kp)}{(lp - hr)} = 0$$

This may be simplified by dividing each term by y/b and multiplying each term by $(lp - hr)$. We then obtain the following equation:

$$e(kr - lq) + f(lp - hr) + g(hq - kp) = 0 \dots \dots \dots (7)$$

This is known as the equation of zone control, as it enables one to determine whether any given face lies in a zone with two known faces or not.

If in equation (7) we substitute $(h + p, k + q, l + r)$ for (efg) we obtain:

$$(h + p)(kr - lq) + (k + q)(lp - hr) + (l + r)(hq - kp) = 0 \dots (8)$$

And also in the same equation if we substitute $(h - p, k - q, l - r)$ for (efg) we obtain:

$$(h - p)(kr - lq) + (k - q)(lp - hr) + (l - r)(hq - kp) = 0 \quad (9)$$

Since we find that the algebraic sum of the products of the left side of both equations (8) and (9) is equal to zero, we have definite proof of the addition and subtraction rule.

In the addition and subtraction of indices, $(mh.mk.ml)$ and $(np.nq.nr)$, where m and n are whole numbers, may be substituted for (hkl) and (pqr) respectively, since the above equations are valid when these substitutions are made. This is obvious, for all parallel planes on the same side of the origin are equivalent.

PREVIOUS USE OF THE ADDITION AND SUBTRACTION
RULE IN GEOMETRICAL CRYSTALLOGRAPHY

The addition of the Miller symbols of two faces of a zone taken index by index to give another face of the zone has been made use of to some extent.

(1) To determine the symbol of a face truncating the edge between two known faces. For example, the symbol of the face truncating the edge between the rhombic dodecahedral faces (110) and (101) is (211) , $(1+1=2, 1+0=1, 0+1=1)$. Similarly, the face truncating the polar edges of the unit rhombohedron $(101\bar{1})$ is $(01\bar{1}2)$. The two faces involved are $(10\bar{1}1)$ and $(\bar{1}101)$. $(1+\bar{1}=0, 0+1=1, \bar{1}+0=\bar{1}, 1+1=2.)$

(2) Another use of the addition rule is to determine the sign to be given to the indices of a face which has been found by cross-multiplying two zone-symbols. Any two zones necessarily meet in two opposite parallel faces. The face obtained depends upon which zone-symbol is placed first in cross-multiplying. If it is known that the face in question lies between two known faces its symbol may be obtained by the addition of indices.

(3) The addition of the indices of the two faces of a zone to give another face of the zone has been used in the law of complication first discovered by Junghann² and independently by Goldschmidt,³ and developed by Goldschmidt, Fedorov,⁴ and Baumhauer.^{5,6}

Very little use apparently has been made of the subtraction of the indices of the faces of a zone to obtain another face of the zone. As a matter of fact this part of the addition and subtraction rule is not generally known. I have seen only two references to it. Story-Maskelyne⁷ says: "Where $\lambda = \mu \pm 1$, the process is simply the addition or subtraction of the indices in the symbols $(h_1k_1l_1)$

² *Pogg. An.*, **151**, 68-95 (1874).

³ *Zeit. f. Kryst. u. Min.*, **28**, 1-35 (1897).

⁴ *Ibid.*, **32**, 446-64 (1900). *Ibid.*, **35**, 25-74 (1902).

⁵ *Ibid.*, **38**, 628-55 (1904).

⁶ DIE NEUE ENTWICKELUNG DER KRISTALLOGRAPHIE, pp. 129-43 (1905).

⁷ *Crystallography*, p. 70, 1895.

and $(h_2k_2l_2)$. Whence, whether we take the sums or differences of the corresponding indices in two symbols, we equally obtain a symbol for a face belonging to the zone."

And Baumhauer⁸ makes the statement: "Die Symbole von zwei solchen Flächen [he is speaking of two faces (hkl) and $(h_1k_1l_1)$ of a zone] werden aber erhalten durch Addition oder Subtraktion der Indices der beiden Ausgangsflächen:

$$h' = h + h_1, \quad k' = k + k_1, \quad l' = l + l_1, \\ \text{und } h'' = h - h_1, \quad k'' = k - k_1, \quad l'' = l - l_1."$$

If the addition of indices of two faces of a zone will give another face of the zone, it necessarily follows that the subtraction of indices will also, since addition and subtraction are inverse operations.

Since (hkl) , (pqr) , and $(h+p.k+q.l+r)$ are in a zone, it is to be noted that the subtraction of the corresponding indices of (hkl) from those of $(h+p.k+q.l+r)$ will give (pqr) , the other face in the zone, and similarly, the subtraction of the indices of (pqr) from the corresponding indices of $(h+p.k+q.l+r)$ gives (hkl) .

ILLUSTRATIONS OF THE ADDITION AND SUBTRACTION RULE

Fig. 2 represents in gnomonic projection two prominent zones of an hypothetical orthorhombic crystal. The expression for the zone $[\bar{1}01:012:210]$ on expansion becomes $[\bar{2}10:\bar{3}11:101:\bar{1}13:012:123:111:432:321:531:210]$. The intermediate faces such as (123) , (432) , (321) , etc., result from the addition of indices thus: $0,1,2+1,1,1=1,2,3$. $1,1,1+2,1,0=3,2,1$. $1,1,1+3,2,1=4,3,2$. And if the faces such as (321) , (432) , (123) , and (111) had been given, the other faces of the zone would result from the subtraction of indices thus: $3,2,1-1,1,1=2,1,0$. $1,2,3-1,1,1=0,1,2$. $1,2,3-3,2,1=\bar{2},0,2=\bar{1},0,1$.

It should be noted that the addition of indices of two faces gives a face which lies *between* the two given faces, while the subtraction of the indices of two faces gives a face which lies *beyond* the two given faces. For example, (432) lies between (111) and (321) and is obtained thus: $(1,1,1+3,2,1=4,3,2)$, but (321) lies beyond (111) and (432) and is obtained thus: $(4,3,2-1,1,1=3,2,1)$.

The other zone indicated in Fig. 2 is the zone $[\bar{2}10:201:011]$ which on expansion by the addition and subtraction rule becomes: $[\bar{2}10:4\bar{1}1:6\bar{1}2:201:413:212:011:\bar{2}32:\bar{2}21:210]$.

⁸ *Fortschritte der Min., Krist., u. Petrog.*, 1, 23 (1911).

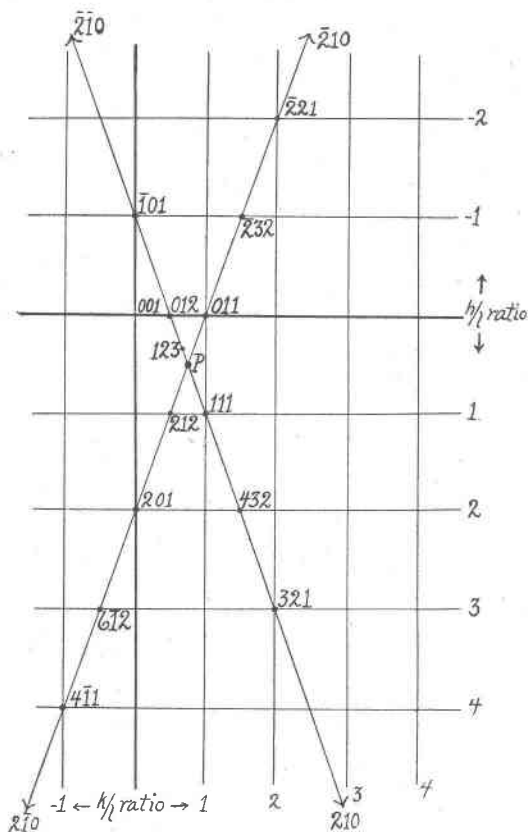


Fig. 2.

Gnomonic projection of two zones $[210:012:\bar{1}01]$ and $[2\bar{1}0:201, 011]$

GENERAL USE OF THE ADDITION AND SUBTRACTION RULE

The addition and subtraction rule may be used instead of the equation of zone control to determine whether a given face lies in a zone with two known faces or not. This method is based upon the fact that for any face (efg) in the zone $[hkl:pqr]$ the following relation must hold: $mh+np=e$, $mk+ng=f$, $ml+nr=g$ where m and n are whole numbers. With the indices $e, f, g, h, k, l, p, q,$ and r given, it is only necessary to solve the above equations for m and n in order to obtain the result. As an example let us take the following: Is the face (135) in the zone $[321:432]$? Assigning m to (321) and n to (432) we have the following equations: $3m+4n$

$=1$; $2m+3n=3$; $m+2n=5$. Solving any two of these equations we find that m is -9 and n is 7 . Then since

$$\text{and} \quad -9 \times 3, 2, 1 = \overline{27} \cdot \overline{18} \cdot \overline{9}$$

$$7 \times 4, 3, 2 = \frac{28 \cdot 21 \cdot 14}{1 \cdot 3 \cdot 5}$$

The addition of these two, index by index, gives us (135), which proves that this face is in the zone $[321:432]$.

In applying this test we may disregard the subtraction part of the rule, for the addition of a negative number is the equivalent of the subtraction of a positive number.

It is also possible by a similar method to determine the indices of a face which lies at the intersection of two zones without determining the indices of the zone.

For example, let us determine the face at the intersection of the zones $[123:321]$ and $[212:221]$ of Fig. 2. The face sought (efg) must satisfy two sets of equations since it lies in each of the two zones. For the first zone the equations are $m+3n=e$, $2m+2n=f$, and $3m+n=g$. For the other zone the equations are $2m'-2n'=e$, $m'+2n'=f$, and $2m'+n'=g$, where m , n , m' , and n' are whole numbers. We then have: $m+3n=2m'-2n'$; $2m+2n=m'+2n'$; and $3m+n=2m'+n'$. Since we have four unknowns and only three equations it is necessary to express one of the unknowns, say n' , as a unit multiplier, $n'/n'=1$. The equations then become:

$$\frac{m}{n'} + \frac{3n}{n'} = \frac{2m'}{n'} - 2, \quad \frac{2m}{n'} + \frac{2n}{n'} = \frac{m'}{n'} + 2, \quad \frac{3m}{n'} + \frac{n}{n'} = \frac{2m'}{n'} + 1.$$

Solving for

$$\frac{m}{n'}, \quad \frac{n}{n'}, \quad \text{and} \quad \frac{m'}{n'}$$

we obtain

$$\frac{m}{n'} = \frac{15}{8}, \quad \frac{n}{n'} = \frac{3}{8}, \quad \frac{m'}{n'} = \frac{20}{8}, \quad \frac{n'}{n'} = \frac{8}{8}.$$

Then, since

$$15 \times 1, 2, 3 = 15 \cdot 30 \cdot 45$$

$$3 \times 3, 2, 1 = 9 \cdot 6 \cdot 3$$

On adding we have
and since

$$24 \cdot 36 \cdot 48 \text{ or } (234)$$

$$20 \times 2, 1, 2 = 40 \cdot 20 \cdot 40$$

$$8 \times 2, 2, 1 = 16 \cdot 16 \cdot 8$$

on adding we have

$$24 \cdot 36 \cdot 48 \text{ or } (234)$$

The face common to the two zones $[123:321]$ and $[212:\bar{2}21]$ is therefore (234) . This may be checked by a graphic determination of the point P in the gnomonic projection of Fig. 2.

It must be admitted that the determination by this method offers no particular advantage over the usual method of cross-multiplication of the two zone-symbols but it is inserted in order to make the discussion more complete and to demonstrate the validity of the addition and subtraction rule.

USE OF THE ADDITION AND SUBTRACTION RULE IN CRYSTAL DRAWING

The addition and subtraction rule is a very useful aid in constructing all crystal drawings in which there is a zone-axis normal to the plane of projection, that is to say in certain orthographic projections. Orthographic projections are usually made from stereographic or gnomonic projections, but the method here outlined enables one to dispense with these projections.

The principle of the method depends upon the fact that the projection of every intersection edge in an orthographic projection of the kind mentioned is parallel to the projection of a possible face in the zone which has its zone-axis normal to the plane of the drawing. Since the face sought always lies beyond, and not between, the two given faces the indices must be subtracted.

As a simple example let us make a plan of a trapezohedral (211) crystal of garnet (see Fig. 3). The outline is easily constructed since the edges $(211:21\bar{1})$ and $(121:12\bar{1})$ are parallel to the projections of the possible faces (210) and (120) respectively. Next the intersection edge between (211) and (112) is found by the subtraction of indices to be parallel to (310) . $2,1,1 \times 2 = 4,2,2$. $4,2,2 - 1,1,2 = 3,1,0$. Similarly, the intersection edge between (121) and (112) is found to be parallel to (130) . $1,2,1 \times 2 = 2,4,2$. $2,4,2 - 1,1,2 = 1,3,0$. Subtraction must be used here since in each case the third index must be equal to zero. The drawing may then be completed.

On all the drawings in this article the direction of the intersection edges is indicated by the symbol of the possible face placed directly over the intersection edge.

Another example is a front elevation of a barite crystal with the forms (110) , (011) , (111) , (001) , (102) , (104) , and (122) .

Fig. 4. With the axial ratio $b:c=1.0:1.313$ for barite, lines representing the various faces of the (001) and (011) forms are first drawn. The other lines of the drawing are either vertical and horizontal lines or are parallel to the projection of possible $(0kl)$, $(0\bar{k}l)$, $(0\bar{k}l)$ or $(0k\bar{l})$ faces. The direction of an edge is found by subtracting indices so that the first index is 0. The projection of

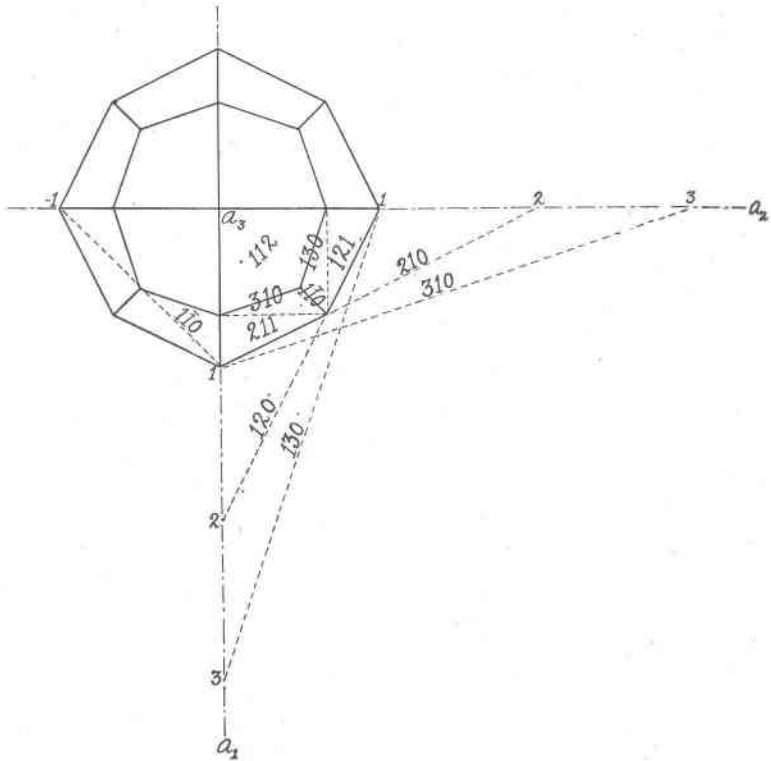


Fig. 3.
Plan of a Garnet Crystal.

the edge $(102:110)$ is $(0\bar{1}2)$. $1,0,2-1,1,0=0,\bar{1},2$. The projection of the edge $(102:111)$ is $(0\bar{1}1)$. $1,0,2-1,1,1=0,\bar{1},1$. The projection of the $(104:111)$ edge is $(0\bar{1}3)$. $1,0,4-1,1,1=0,\bar{1},3$. The projection of the $(122:111)$ edge is (011) . $1,2,2-1,1,1=0,1,1$. The projection of the $(122:110)$ edge is (012) . $1,2,2-1,1,0=0,1,2$. With the data thus found the drawing may easily be constructed.

Still another example is shown in Fig. 5, which is a side elevation of an augite crystal with the forms (100), (110), (010), and $(\bar{1}11)$. This side elevation may be constructed from a stereographic or gnomonic projection or from a plan and a supplementary projection made from the interfacial angles $(\bar{1}11:\bar{1}11)$ and $(\bar{1}11:010)$ as has been pointed out by the writer in a recent article.⁹ In the last case the procedure would be similar to that given for gypsum.¹⁰

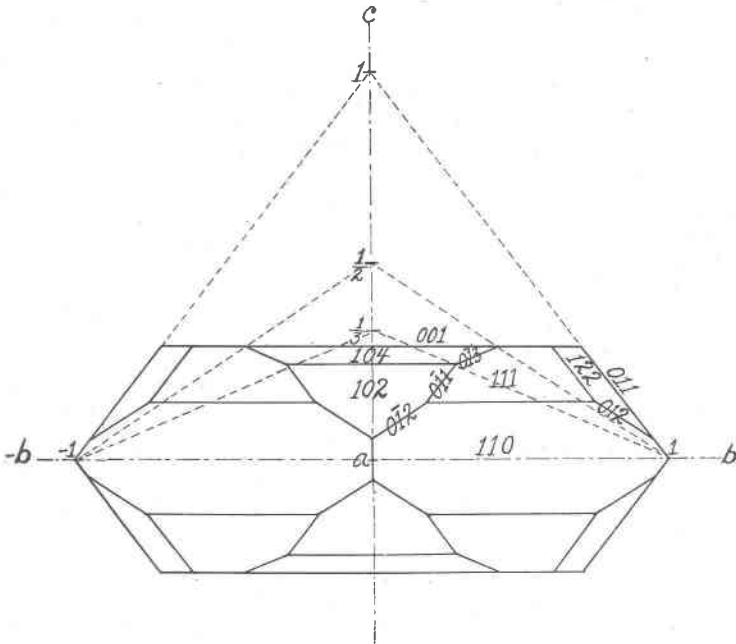


Fig. 4.
Front Elevation of a Barite Crystal.

A more direct method of constructing the side elevation of augite involves the axial ratio $a:c$ and use of the addition and subtraction rule. The outline is made by laying off the axial ratio $a:c = 1.09:0.59$ and $\beta(a:c) = 74^\circ 10'$. The edge $(\bar{1}11:\bar{1}11)$ is parallel to the projection of the possible face $(\bar{1}01)$. The projection of the edge $(110:\bar{1}11)$ is found as follows: $\bar{1}, 1, 1-1, 1, 0 = \bar{2}, 0, 1$, and the projection of the edge $(\bar{1}11:\bar{1}10)$ is found as follows: $\bar{1}, 1, 1 - \bar{1}, 1, 0$

⁹ *Am. Mineral.*, 8, 19-31 (1923).

¹⁰ *Ibid.* See Fig. 4, p. 25.

$=0,0,1$. Subtraction is used in both cases since the second index in each case must be zero. Lines are drawn parallel to the traces of the $(\bar{2}01)$ and (001) faces and the drawing may be completed.

A more complicated example is shown in Fig. 6. Here we have an angular elevation of a topaz crystal with the forms: (110) , (120) , (010) , (021) , (041) , (223) , (111) , and (221) . In the three preceding examples the zone-axis normal to the plane of projection is the c -axis, the a -axis, and the b -axis, so that the direction

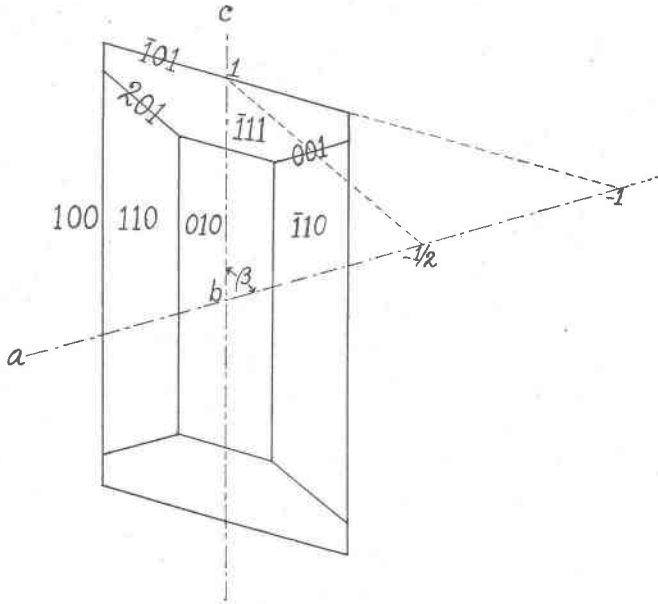


Fig. 5.
Side Elevation of an Augite Crystal.

of the intersection edges of the drawings are parallel to possible $(hk0)$, $(0kl)$ and $(h0l)$ faces respectively. In the drawing of Fig. 6 we have a different case. Here the plane of projection is a plane normal to the zone-axis $[1\bar{1}0]$. The lines of the drawing are parallel to traces of possible (hhl) and $(\bar{h}hl)$ faces including (110) and (001) . In each case the multipliers must be chosen so that the subtraction of indices will give a symbol in which the first index is the same as the second index.

After the bounding faces (110), (221), (111); (223), $\bar{2}\bar{2}3$, $\bar{1}\bar{1}1$, $\bar{2}\bar{2}1$, and $\bar{1}\bar{1}0$ and the edge (021:0 $\bar{2}$ 1) are projected, the direction of the intersection edges is obtained as follows: The direction of the (021:120) edge is $\bar{2}\bar{2}1$. $1,2,0 \times 2 = 2,4,0$. $0,2,1 - 2,4,0 = \bar{2},\bar{2},1$. The direction of the (041:120) edge is $\bar{4}\bar{4}1$. $1,2,0 \times 4 = 4,8,0$. $0,4,1 - 4,8,0 = \bar{4},\bar{4},1$. The direction of the (041: $\bar{1}$ 20) edge is $\bar{4}\bar{4}3$. $0,4,1 \times 3 = 0,12,3$. $\bar{1},2,0 \times 4 = \bar{4},8,0$. $0,12,3 - \bar{4},8,0 = 4,4,3$. The

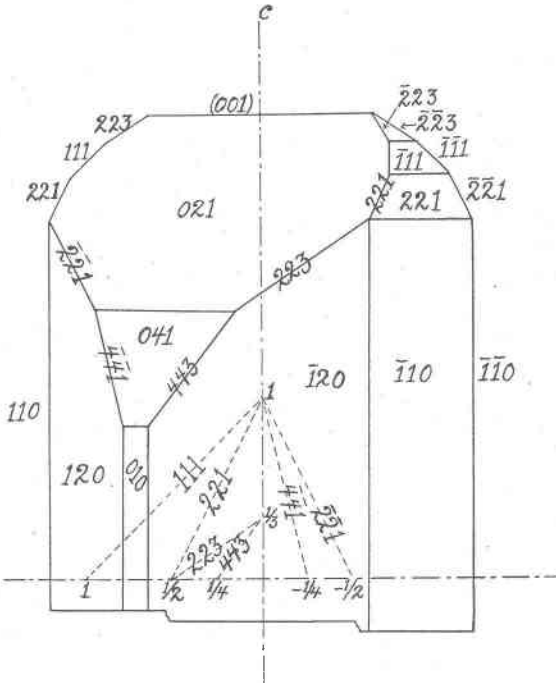


Fig. 6.
Angular Elevation of a Topaz Crystal.

direction of the (021: $\bar{1}$ 20) edge is $\bar{2}\bar{2}3$. $0,2,1 \times 3 = 0,6,3$. $\bar{1},2,0 \times 2 = \bar{2},4,0$. $0,6,3 - \bar{2},4,0 = 2,2,3$. And in the same way the direction of the other edges is found. The drawing may then be completed.

In conclusion the opinion is expressed that the addition and subtraction rule is a useful aid in the problems encountered in the delineation and calculation of crystals.