THE AMERICAN MINERALOGIST

JOURNAL OF THE MINERALOGICAL SOCIETY OF AMERICA

Vol. 10. AUGUST, 1925 No. 8

FRIEDEL'S LAW OF RATIONAL SYMMETRIC INTERCEPTS[1]

WITH BIBLIOGRAPHY OF IRRATIONAL THREE-FOLD AXIS OF SYMMETRY

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There are two generally accepted fundamental laws of geometrical crystallography: (1) the law of constancy of interfacial angles proved for quartz by Steno in 1669, but first established as a general law by Rome de l'Isle in 1772 after the invention of the contact goniometer; (2) the law of simple rational indices, which may also be expressed as the law of zones, discovered by Haüy in 1784. As has been pointed out by Friedel,[2] Bravais' law of maximum reticulate density is a more precise form of Haüy's law of simple rational indices.

The so-called law of symmetry proposed by Haüy in 1815, which may be expressed thus: "Similar parts of a crystal are similarly modified"[3] does not deserve the name of a law. It is a statement of no particular significance, for symmetry is determined by similar modifications.

There is, however, a third law of geometrical crystallography, viz., the law of rational symmetric intercepts[4] (or parameters), which was formulated by G. Friedel in 1905. Apparently this contribution of Friedel has been overlooked or ignored. As far as can be learned the writer (28) is the only one besides Friedel himself who has referred to the law of rational symmetric intercepts in print. The object of this note is to call attention to Friedel's work in order that it may receive the attention that it deserves.

[1] Paper presented at the annual meeting of the Mineralogical Society of America, Washington, D. C., Dec. 29, 1923.

[2] *Bull. Soc. Franc. Min.*, **30**, 326-455 (1907).

[3] Miers, Mineralogy, p. 15, London, **1902**.

[4] Friedel uses the term parameter instead of intercept, but on account of the ambiguity in the use of parameter the writer has substituted intercept for it.

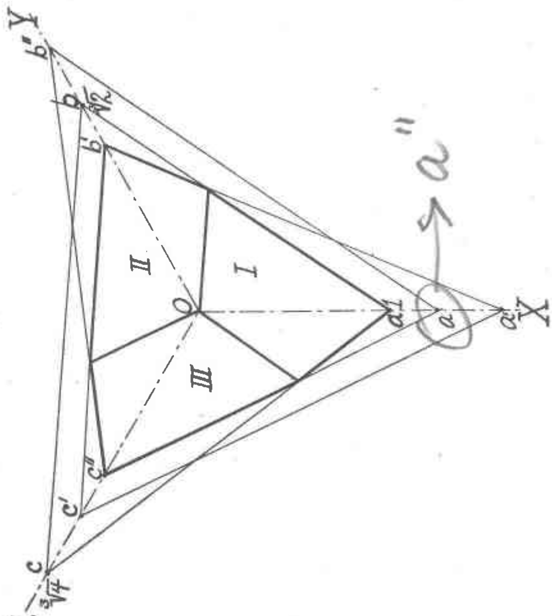

Fig. 1

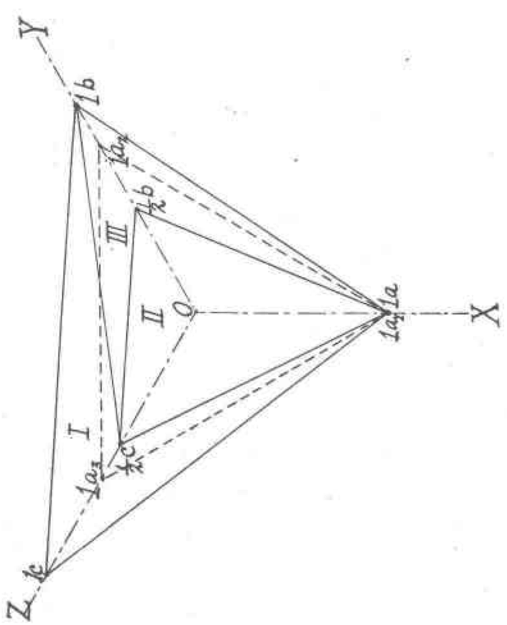

Fig. 2

Diagrams to illustrate irrational three-fold axis of symmetry and Friedel's law of rational symmetric intercepts.

As was first pointed out by Gadolin (1) in a classic paper read in 1867 and published in 1871, an axis of ternary or threefold symmetry in a crystalloid polyhedron is not necessarily a zone-axis and is not necessarily normal to a possible face or plane.[5] In other words a polyhedron may have planes with simple rational indices and irrational symmetric intercepts (or parameters). This is shown graphically in Figs. 1 and 2. Since the time of Gadolin the "irrational ternary axis" has been the subject of much discussion by crystallographers. Prominent among those who have taken part in the discussion are Fedorov (3, 5, 7, 9, 12, 13), Hecht (4, 6, 8, 14), Souza-Brandão (10, 18), Viola (11, 15, 17), and Friedel (23, 25). The irrational three-fold axis of symmetry is mentioned in the treatises or text-books of Lewis (19), Hilton (21), Sommerfeldt (24), Bouasse (26) and Friedel (27), but is entirely ignored by most authors. While perhaps it may appropriately be omitted from elementary text-books, it should certainly find a place in the more advanced text-books and treatises.

It is not my purpose to enter into a detailed discussion of the controversy relative to the possibility of the irrational three-fold symmetry axis. A bibliography of the subject is appended to this article for those who wish to look into the matter at length. The essential features of the question may be presented very simply by means of diagrams. The lack of suitable drawings to illustrate the subject under discussion probably accounts for a general lack of understanding of the principles involved.

In Fig. 1 *OX*, *OY*, and *OZ* represent in orthographic projection three axes of reference equally inclined to each other at either right or oblique angles. A line normal to the plane of the drawing through the origin, *O*, represents a three-fold axis of symmetry. The plane I with the intercepts a, b, and c on counter-clockwise rotation of 120° will give rise to the plane II with the intercepts a', b' and c'; and this plane on a further rotation of 120° will give rise to the plane III with intercepts a'', b'' and c''. These

[5] The exceptional character of a three-fold axis of symmetry has been brought out by Lewis (19- p. 114). The proof that an axis of symmetry of even degree is a possible zone-axis and is normal to a possible face depends upon the fact that two similar (homologous) zone-axes are coplanar with the axis, and in the case of an axis of odd degree, $n>3$, the proof depends upon the fact that two dissimilar (non-homologous) edges are coplanar with the axis. "Here the homologous zone-axes form the edges of a trigonal pyramid, and no pair of them lies in a plane containing the triad axis. Nor can an auxiliary pyramid be formed by the edges of alternate faces."

three planes meet to form a trigonal pyramid as indicated in the drawing. Now if the intercepts of the plane I are chosen so as to be in the ratio of $a : b : c = 1 : \sqrt[3]{2} : \sqrt[3]{4}$,[6] the intercepts of the plane II are $a' : b' : c' = \sqrt[3]{4} : 1 : \sqrt[3]{2}$, and the intercepts of the plane III, $a'' : b'' : c'' = \sqrt[3]{2} : \sqrt[3]{4} : 1$. Here the intercepts on three symmetric axes of reference are irrational. We may take one of the planes as a standard or unit face and express the other planes in terms of our standard as is customary in crystallography. Let us select plane I as our (111) face. We may shift the planes II and III parallel to themselves so that they cut the OX-axis at the point a. Then all three planes intersect the OX-axis at a common point. The result of such shifting is indicated in Fig. 2. The new intercepts of plane II are $a : b : c = \frac{\sqrt[3]{4}}{\sqrt[3]{4}} : \frac{1}{\sqrt[3]{4}} : \frac{\sqrt[3]{2}}{\sqrt[3]{4}}$ which is obtained by dividing each intercept by $\sqrt[3]{4}$. Similarly, the new intercepts of plane III are $a : b : c = \frac{\sqrt[3]{2}}{\sqrt[3]{2}} : \frac{\sqrt[3]{4}}{\sqrt[3]{2}} : \frac{1}{\sqrt[3]{2}}$, obtained by dividing each intercept by $\sqrt[3]{2}$.

Since we selected plane I as our standard or unit face, it may be expressed by the Weiss symbol $1a : 1b : 1c$ or the Miller symbol (111). Plane II expressed in terms of the unit face is $1a : \frac{1}{2}b : \frac{1}{2}c$ or (122). Plane III expressed in terms of the unit face is $1a : 1b : \frac{1}{2}c$ or (112). The relations are shown in the following tabulation:

		Weiss Miller
Plane I	$Oa : Ob : Oc = 1 : \sqrt[3]{2} : \sqrt[3]{4} = 1 : \sqrt[3]{2} : \sqrt[3]{4} =$	$1a : 1b : 1c = (111)$
Plane II	$Oa' : Ob' : Oc' = \sqrt[3]{4} : 1 : \sqrt[3]{2} = \frac{\sqrt[3]{4}}{\sqrt[3]{4}} : \frac{1}{\sqrt[3]{4}} : \frac{\sqrt[3]{4}}{\sqrt[3]{4}} =$	$1a : \frac{1}{2}b : \frac{1}{2}c = (122)$
Plane III	$Oa'' : Ob'' : Oc'' = \sqrt[3]{2} : \sqrt[3]{4} : 1 = \frac{\sqrt[3]{2}}{\sqrt[3]{2}} : \frac{\sqrt[3]{4}}{\sqrt[3]{2}} : \frac{1}{\sqrt[3]{2}} =$	$1a : 1b : \frac{1}{2}c = (112)$

The relations indicated may be proved either mathematically or graphically as is done in the drawings. If we substitute for the values $\sqrt[3]{2}$ and $\sqrt[3]{4}$, 1.2599 and 1.5874 respectively, it is easy to prove that the above expressions are correct.

Here is actual demonstration that in a crystalloid polyhedron it is mathematically possible to have rational indices of the faces and irrational axial ratios for the symmetric axes of reference.

[6] The intercepts in general may be $1 : \sqrt[3]{n} : \sqrt[3]{n^2}$ where n is any small integer. In order to construct a diagram it is necessary to use a particular value for n; in Figs. 1 and 2 it has the value 2.

This is possible with the four following types of symmetry according to Friedel (23, 25): A_3, trigonal pyramidal class; $\overline{AP}_6(A_3)$ (C), rhombohedral class; $4A_3.3A_2$, tetartoidal class; $4A_3(4\overline{AP}_6).3A_2.3P(C)$, diploidal class. Friedel suggested the possibility of two additional crystal systems for these four divisions. The first two constitute a "système ternaire irrational" and the other two, a "système cubique irrational." Crystals belonging to the "irrational ternary system," if such exist, would not possess either prisms, pinacoid, or pedion. The only forms possible would be trigonal pyramids, and rhombohedrons. In the "irrational cubic system" neither octahedrons, tetrahedrons, rhombic dodecahedrons, trapezohedrons, or trisoctahedrons could occur. The only forms possible would be diploids, pyritohedrons, tetartoids, and cubes.

For example in Fig. 2 the face shown by dotted lines is an impossible face with the axes of reference postulated. But it has been found that the pedion, pinacoid, and prism are found on crystals with the symmetry A_3 and $\overline{AP}_6(A_3)(C)$ and also that the tetrahedron or octahedron, rhombic dodecahedron, trapezohedron, and trisoctahedron are common on crystals belonging to the tetartoidal and diploidal classes.

In other words in all crystals with either a single axis of three-fold symmetry or four axes of three-fold symmetry the parameters or intercepts of a unit face on three axes of reference symmetrical to a three-fold axis are equal or rational. In Fig. 2 for example, the three axes of reference $a_1 : a_2 : a_3$ are equal and interchangeable. This is a fact of observation and Friedel (23) cleared up the whole matter by formulating the following law of rational symmetric parameters: "Two symmetric edges of a crystal have the same parameter." It is only fair to state that this law was hinted at by Viola (17) in 1897 and by Lewis (19) in 1899, but Friedel deserves the main credit for its recognition. His paper entitled "Sur les bases expérimentales de l'hypothèse reticulaire" will become, I believe, one of the classics of geometrical crystallography.

This empirical law of observation practically proved the existence of the space lattice in 1905, seven years before direct proof of the space lattice was furnished by the work of Laue and his associates in 1912.

This important contribution of Friedel's, brilliant in its simplicity, has been overlooked by nearly all crystallographers. The object of this note is to call attention to its importance in the development of the theory of crystal structure.

As I conceive it, there are three stages in the history of the science of crystal structure (leptonology of Rinne). (1) The mathematical stage beginning with the work of Bravais in 1849 and culminating in that of Fedorov, Schoenflies, and Barlow (1890-1894). (2) The crystallographical stage in which Friedel (1905) takes the most prominent part. And (3) the experimental stage beginning with the work of Friedrich and Knipping on sphalerite in 1912 and continuing up to the present.

BIBLIOGRAPHY

Irrational Three-Fold Axis of Symmetry

1. Gadolin, A., Mémoire sur la déduction d'un seul principe de tous les systèmes cristallographiques. *Acta Societatis Scientiarum Fennicae*, Vol. **IX,** pp. 48-51, 1871. (Reprinted in Ostwald's Klassiker der Exakten Wissenschaften, No. 75, pp. 57-60, Leipzig, **1896**.)
2. Liebisch, T., Geometrische Krystallographie, pp. 195-196, *Leipzig*, **1881**.
3. Fedorov, E. S., Zwei Krystallographische Notizen. *Verhandl. d. Russ. Min. Gesellschaft, St. Petersburg* (2), Vol. **25,** pp. 53-58, 1889. (Abstract, *Zeit. f. Kryst. u. Min.*, Vol. **17,** pp. 617-618, 1890.)
4. Hecht, B., Beiträge zur geometrischen Krystallographie. *Nachr. Königl. Gesellschaft d. Wissenschaften, Göttingen,* **1892,** pp. 245-246.
5. Fedorov, E. S., Die Determinanten und das Doppelverhältniss im Gebiete der Krystallographie. *Zeit. f. Kryst. u. Min.*, Vol. **21,** p. 700, 1893.
6. Hecht, B., Bemerkung zu dem Satze, nach welchem Symmetrieaxen immer mögliche Krystallkanten sein sollen. *Neues Jahrb. f. Min., Geol., u. Pal.*, **1893,** II Band, pp. 173-174.
7. Fedorov, E. S., Noch ein Wort über das Satz, nach welchem Symmetrieaxen immer mögliche Krystallkanten sein sollen. *Neues Jahrb. f. Min., Geol., u. Pal.*, **1894,** I Band, pp. 199-200.
8. Hecht, B., Zweite Bemerkung zu dem Sätze, nach welchem Symmetrieaxen, immer mögliche Krystallkanten sein sollen. *Ibid.*, **1894,** I Band, pp. 278-280.
9. Fedorov, E. S., Das Grundgesetz der Krystallographie. *Zeit. f. Kryst. u. Min.*, Vol. **23,** pp. 108-112, 1894.
10. Souza-Brandão, V., Ueber die Rationalität einer dreizähligen Symmetrieaxe. *Ibid.*, Vol. **23,** pp. 249-258, 1894.
11. Viola, C., Ein Wort über den Hauptsatz der Symmetrie der Krystalle. *Ibid.*, Vol. **23,** pp. 466-467, 1894.
12. Fedorov, E. S., Theorie der Krystallstructur. *Zeit. f. Kryst. u. Min.*, Vol. **24,** pp. 244-247, 1895.
13. Fedorov, E. S., Ueber die Bedeutung der die Krystallflächen-complexe bestimmenden Parameter (Elemente eines Krystalls). *Zeit. f. Kryst. u. Min.*, Vol. **24,** pp. 607-610, 1895.
14. Hecht, B., Ueber den Beweis des Satzes von der Rationalität einer dreizähligen Symmetrieaxe. *Neues Jahrb. f. Min., Geol., u. Pal.*, **1895,** II Band, pp. 248-252.

15. Viola, C., Ueber geometrische Ableitung in der Krystallographie. *Zeit. f. Kryst. u. Min.*, Vol. **26**, pp. 128-129, 1896.
16. v. Lang, V., Notiz zur trigonalen Symmetrie. *Ibid.*, Vol. **27**, pp. 91-92, 1897.
17. Viola, C., Beweis der Rationalität einer dreizähligen Symmetrieaxe. *Ibid.*, Vol. **27**, pp. 399-405, 1897.
18. Souza-Brandão, V., Noch ein Wort über die bedingte Rationalität einer dreizähligen Symmetrieaxe. *Ibid.*, Vol. **27**, pp. 545-555, 1897.
19. Lewis, W. J., A Treatise on Crystallography, *Cambridge*, **1899**, pp. 114-115.
20. Barlow, W., On Crystal Symmetry. *Phil. Mag.*, (6th series), Vol. **I**, pp. 1-36, 1901.
21. Hilton, H., Mathematical Crystallography and the Theory of Groups of Movements. *Oxford*, **1903**, pp. 39-41, p. 144.
22. Sommerfeldt, E., Beitrag zur Symmetrie trigonaler Punktsysteme, *Centralblatt f. Min., Geol., u. Pal.*, **1904**, pp. 25-27.
23. Friedel, G., Sur les bases experimentales de l'hypothèse réticulaire. *Bull. Soc. Franc. Min.*, Vol. **28**, pp. 95-150, 1905.
24. Sommerfeldt, E., Physikalische Kristallographie, *Leipzig*, **1907**, pp. 113-117.
25. Friedel, G., Réponse à M. Mügge au sujet des axes ternaires irrationels. *Bull. Soc. Franc. Min.*, Vol. **30**, pp. 6-17, 1907.
26. Bouasse, H. P. M., Cours de Physique, 6me Partie (Éude des Symétries) pp. 67-69, *Paris*, **1909**.
27. Friedel, G., Leçons de Cristallographie, *Paris*, **1911**, pp. 109-113.
28. Rogers, A. F., Introduction to the Study of Minerals and Rocks, 2nd edition, pp. 74, 134, 139, *New York*, **1921**.

ZIRCON, A CONTACT METAMORPHIC MINERAL IN THE PEND OREILLE DISTRICT, IDAHO[1]

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INTRODUCTION

The purpose of this paper is to call attention to a rather widespread transfer of zirconium from some intrusive rocks into invaded sedimentary rocks in the Pend Oreille mining district, Idaho. This district, an area 15 by 20 miles, was recently studied by a party of the U. S. Geological Survey under Dr. Edward Sampson. The contact metamorphism was investigated in detail, and in the course of this work the prevalence of very minute zircon crystals in the most metamorphosed rocks turned the attention of the writer to a special study of this problem. Conclusions were reached only after a study of over a hundred thin sections and the examination of the heavy residues of more than thirty rock samples. In addition, Professor R. A. Daly of Harvard University very kindly loaned a number of thin sections of similar rocks from the Cana-

[1] Published by permission of the Director of the U. S. Geological Survey.